ON FACTORISATION OF A CLASS OF SCHRÖDINGER OPERATORS

ARI LAPTEV

ABSTRACT. The aim of this paper is to find inequalities for $3/2$ moments of the negative eigenvalues of Schrödinger operators on half-line that have a 'Hardy term" by using the commutator method.

1. INTRODUCTION

Let $V \geq 0$ be a real-valued function on $\mathbb{R}$ decreasing at infinity rapidly enough. Then the negative spectrum of the Schrödinger operator $-\frac{-d^2}{dx^2} - V$ in $L^2(\mathbb{R})$ consists of discrete negative eigenvalues $\{-\lambda_k\}$ of finite multiplicities. The celebrated Lieb-Thirring inequality states that

$$\sum_k \lambda^\gamma \leq L_\gamma \int_{\mathbb{R}} V(x)^{\gamma+1/2}\, dx.$$

This inequality holds with a constant L_γ independent of V if and only if $\gamma \geq 1/2$. In the non-critical case $\gamma > 1/2$ this result was obtained by E.Lieb and W.Thirring in [LT], where the authors also found the sharp constant L_γ for $\gamma = 3/2$ and then in [AzL] for all $\gamma \geq 3/2$. Later the result on sharp constants for the case $\gamma \geq 3/2$ was extended to the multi-dimensional case in [LW1], [LW2] .

The inequality with a constant independent of V at the critical case $\gamma = 1/2$ was established by T.Weidl in [W] and later Hundertmark-Lieb-Thomas [HLT] found the sharp value of the constant $L_{1/2}$ that equals $1/2$ (see also [HLW]). The sharp constants for $1/2 < \gamma < 3/2$ are still unknown. Some progress on the values of the best constants was given in the paper [DLL] with recent improvements in [FHJN].

The aim of this paper is to prove the following result concerning the so-called Hardy-Lieb-Thirring inequality in the case $\gamma = 3/2$.

Let $\alpha \geq 1/2$ and let $\mathcal{H}_\alpha$ be the differential operator in $L^2(0,\infty)$ with Dirichlet boundary at conditions at zero defined by

$$\mathcal{H}_\alpha = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2} - V, \tag{1.1}$$

Let λ_k and $\{u_k\}$ be the eigenvalues and orthonormal eigenfunctions of the operator $\mathcal{H}_\alpha$. Denote

$$D_\alpha = \frac{d}{dx} - \frac{\alpha}{x}.$$

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Theorem 1.1. *Let $\alpha \geq 1/2$ and assume that $V \in L^2(0,\infty)$. Then for the negative eigenvalues $\{-\lambda_k\}_{k=1}^n$ of the operator $\mathcal{H}_\alpha$ acting in $L^2(0,\infty)$ with the Dirichlet boundary condition at zero we have the following inequality*

$$\sum_{k=1}^n \lambda_k^{3/2} \leq \frac{3}{8}\int_0^\infty V^2(x)\,dx - \sum_{k=1}^n(\alpha-1+k)\int_0^\infty \frac{1}{x^2}\,F_k^2(x)\,dx, \quad n\in\mathbb{N}.$$

Here

$$F_1 = \frac{D_\alpha u_1}{u_1}, \qquad F_k = \frac{v_k'}{v_k},$$

where

$$v_k = \Pi_{j=1}^{k-1}\,(D_{\alpha-1+j} - F_j)u_k, \; k\geq 2.$$

A weaker version of this statement is the following:

Corollary 1.2. *For the negative eigenvalues of the operator* (1.1) *we have*

$$\sum_k \lambda_k^{3/2} \leq \frac{3}{8}\int_0^\infty V^2(x)\,dx. \tag{1.2}$$

In order to prove Theorem 1.1 we use a factorisation of the operator $\mathcal{H}_\alpha$ with two differential operators of order one. Note that after the first swap of two factors (Darboux transform), the original operator $\mathcal{H}_\alpha$ becomes

$$\mathcal{H}_{\alpha+1} = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha+1)}{x^2} - \widetilde{V}, \tag{1.3}$$

also with Dirichlet boundary conditions and with some new potential $\widetilde{V}$. The inequality $\alpha\geq 1/2$ implies $\alpha(\alpha+1)>0$ and thus by using the variational principal we can omit the "Hardy term" in (1.3). This allows us to extend this operator to functions equal to zero on $(-\infty,0)$. Applying known results for Schrödinger operators in $L^2(-\infty,\infty)$ we obtain:

Theorem 1.3. *Let $\alpha\geq 1/2$ and assume that $V\in L^2(0,\infty)$. Then for the operator* (1.1) *we have*

$$\frac{8}{3}\lambda_1^{3/2}(\mathcal{H}_\alpha) + \frac{16}{3}\sum_{k=2}^\infty \lambda_k^{3/2}(\mathcal{H}_\alpha) \leq \int_0^\infty V^2(x)\,dx - 4\alpha\int_0^\infty \frac{1}{x^2}\,F_1^2(x)\,dx, \tag{1.4}$$

where $F_1 = D_\alpha u_1/u_1$ and u_1 is the eigenfunction of $\mathcal{H}_\alpha$ corresponding to the first eigenvalue of $-\lambda_1$.

In particular, we can also state the result for the critical case $\alpha = 1/2$ (compare with Corollary 1.2).

Corollary 1.4. *If $\alpha = 1/2$. Then under the conditions of Theorem 1.1 we have*

$$\frac{8}{3}\lambda_1^{3/2}(\mathcal{H}_{1/2}) + \frac{16}{3}\sum_{k=2}^\infty \lambda_k^{3/2}(\mathcal{H}_{1/2}) \leq \int_0^\infty V^2(x)\,dx - 2\int_0^\infty \frac{1}{x^2}\,F_1^2(x)\,dx. \tag{1.5}$$

The fact that the inequality

$$\sum_k \lambda^\gamma(\mathcal{H}_\alpha) \le C_{\alpha,\gamma} \int_0^\infty V^{\gamma+1/2}(x)\, dx$$

with $\alpha \ge 1/2$ and $\gamma \ge 1/2$ holds with a constant $C_{\alpha,\gamma}$ independent of V was discovered by T. Ekholm and R. Frank in [EF1], [EF2], see also [F] and [FLS]. However, it was clear that the constants in these inequalities were far from being sharp.

Conjecture. We conjecture that the constant $3/8$ obtained in Theorem 1.1 and in Corollary 1.2 is sharp.

In this paper we use an approach which was originally suggested by R.Benguria and M.Loss in [BL], where the authors obtained Lieb-Thirring inequalities for a Schrödinger operator in $L^2(\mathbb{R})$ with a matrix-valued potential using the Darboux transform. This approach was later used in [ELU] for obtaining Lieb-Thirring inequalities for operators with Robin boundary conditions, see also [Sch].

2. Proof of Theorem 1.1

Let the operator

$$H_\alpha = D_\alpha^* D_\alpha = \Big(-\frac{d}{dx} - \frac{\alpha}{x}\Big)\Big(\frac{d}{dx} - \frac{\alpha}{x}\Big) = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2}.$$

be acting in $L^2(\mathbb{R}_+)$ on functions with Dirichlet boundary conditions at zero. Let $\lambda > 0$. Consider the equation

$$\left(-\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2}\right)\varphi = -\lambda\varphi.$$

After the substitution $\varphi(x) = \sqrt{x}\psi(x)$ this equation becomes

$$-\sqrt{x}\psi''(x) - \frac{1}{\sqrt{x}}\psi'(x) + \frac{1}{4}\,\frac{1}{x^{3/2}}\psi(x) + \frac{\alpha(\alpha-1)}{x^{3/2}} = -\lambda\,\sqrt{x}\,\psi(x),$$

or

$$\psi'' + \frac{1}{x}\psi' - \frac{(\alpha-1/2)^2}{x^2}\,\psi = \lambda\psi,$$

which is the standard form of the Bessel equation. Since $\lambda > 0$ the two solutions of this equation are modified Bessel functions $I_\nu(\sqrt{\lambda}x)$ and $K_\nu(\sqrt{\lambda}x)$, with $\nu = (\alpha - 1/2)$. We have

$$I_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^\infty \frac{1}{\Gamma(k+1)\,\Gamma(\nu+k+1)}\Big(\frac{z}{4}\Big)^{2k} =$$

$$\frac{1}{\Gamma(1+\nu)}\,\Big(\frac{z}{2}\Big)^\nu\left\{1 + \frac{(z/2)^2}{1(1+\nu)}\left(1 + \frac{(z/2)^2}{2(2+\nu)}\left(1 + \frac{(z/2)^2}{2(2+\nu)}(1+\dots\Big)\right)\right\}.$$

In particular, if $\nu = n = 0, 1, 2, \dots$, then we have

$$I_n(z) = \frac{z^n}{n!}\left[1 + \frac{(z/2)^2}{1(1+n)}\left(1 + \frac{(z/2)^2}{2(2+n)}\left(1 + \frac{(z/2)^2}{2(2+n)}(1+\dots)\right)\right)\right].$$

Moreover,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left\{1 + \frac{\mu}{8z} + \frac{(\mu-1)(\mu-9)}{2(8z)^2} + \dots\right), \quad \text{as} \quad z \to \infty.$$

where $\mu = 4\nu^2$, (see [AS], page 378).

For the operator $\mathcal{H}_\alpha = H_\alpha - V$ we consider the lowest eigenvalue $-\lambda_1$ and the corresponding eigenfunction u_1

$$\mathcal{H}_\alpha u_1 = H_\alpha u_1 - V u_1 = -\lambda_1 u_1.$$

In order to prove Theorem 1.1 it is enough to assume that $V \in C_0^\infty(\mathbb{R}_+)$. Then

$$u_1(x) = \sqrt{x} I_\nu(\sqrt{\lambda_1} x), \quad \text{as} \quad x \to 0,$$

and

$$u_1(x) = \sqrt{x} K_\nu(\sqrt{\lambda_1} x) \quad \text{as} \quad x \to \infty.$$

Therefore

$$u_1(x) = \begin{cases} \frac{\lambda_1^{\nu/2}}{\Gamma(1+\nu)}\, x^{\nu+1/2}\left(1 + \frac{\lambda_1 x^2}{4(1+\nu)} + O(x^4)\right), & \text{as } x \to 0, \\ \sqrt{\frac{\pi}{2\sqrt{\lambda_1}}}\, e^{-\sqrt{\lambda_1}x}(1 + O(x^{-1}), & \text{as } x \to \infty. \end{cases} \tag{2.1}$$

Let us introduce

$$F_1 = \frac{D_\alpha u_1}{u_1}.$$

Then using (2.1) and the equality $\nu + 1/2 = \alpha$ we find

$$F_1(x) = \begin{cases} \frac{\lambda_1(\nu+5/2)}{4(1+\nu)}\, x + O(x^3), & \text{as } x \to 0. \\ -\sqrt{\lambda_1} + O(1/x), & \text{as } x \to \infty. \end{cases} \tag{2.2}$$

Note a useful identity

$$\begin{aligned} F_1'(x) = & \left(\frac{u_1'(x)}{u_1(x)} - \frac{\alpha}{x}\right)' \\ = & \frac{u_1''(x)}{u_1(x)} - \left(\frac{u_1'(x)}{u_1(x)}\right)^2 + \frac{\alpha}{x^2} \\ = & -V(x) + \lambda_1 + \frac{\alpha^2}{x^2} - \left(\frac{u_1'(x)}{u_1(x)}\right)^2 \\ = & -V(x) + \lambda_1 - F_1^2(x) + 2\frac{\alpha^2}{x^2} - 2\frac{\alpha}{x}\,\frac{u_1'(x)}{u_1(x)} \\ = & -V(x) + \lambda_1 - F_1^2(x) - 2\frac{\alpha}{x}\,F(x). \end{aligned}$$

Elementary computations imply

$$A(\alpha) := (D_\alpha^* - F_1)(D_\alpha - F_1) = \mathcal{H}_\alpha + \lambda_1. \tag{2.3}$$

Similar computations lead us to the equation

$$A_1(\alpha) = (D_\alpha - F_1)(D_\alpha^* - F_1) = \mathcal{H}_{\alpha+1} + \lambda_1 - 2F_1'. \tag{2.4}$$

Both equations will be justified later, see Appendix.

The domain $\operatorname{dom} A_1(\alpha)$ of the new of operator $A_1(\alpha)$ is defined by $\{v : v = (D_\alpha - F_1)\, u\}$, where $u \in \operatorname{dom} A(\alpha) = \operatorname{dom} \mathcal{H}_\alpha$. Note that

$$v(0) = \lim_{x\to+0} (D_\alpha - F_1)\, u(x) = \lim_{x\to+0} \left(\frac{d}{dx} - \frac{u_1'(x)}{u_1(x)} \right) u(x) = 0.$$

Therefore both $A_1(\alpha)$ and $\mathcal{H}_{\alpha+1}$ are differential operators with Dirichlet boundary conditions.

The operators $A(\alpha)$ and $A_1(\alpha)$ have the same spectrum with possible exception of the zero eigenvalue. Clearly the kernel of $A(\alpha)$ coincides with the eigenfunction u_1 of the operator$\mathcal{H}_\alpha$. However, the operator $A_1(\alpha)$ does not have trivial eigenvalues. Indeed, if it is the case, then the corresponding eigenfunction would satisfy the identity

$$(D_\alpha^* - F_1)u = -u' - \frac{\alpha}{x}u - F_1 = 0,$$

which is impossible as its asymptotic behaviour at infinity is

$$u(x) \sim x^{-\alpha} e^{\sqrt{\lambda_1} x} \notin L^2(0,\infty).$$

Substituting the expression for F_1' obtained above we have

$$\begin{aligned} \int_0^\infty (V + 2F_1')^2\, dx &= \int_0^\infty V^2\, dx + 4 \int_0^\infty (V + F_1')F_1'\, dx \\ &= \int_0^\infty V^2\, dx + 4 \int_0^\infty \Big(\lambda_1 - F_1^2 - 2\frac{\alpha}{x}\, F_1(x) \Big)\, F_1'\, dx. \end{aligned}$$

Using the values of F at zero and infinity (2.2) find

$$\begin{aligned} \lambda_1 \int_0^\infty F_1'(x)\, dx &= -\, \lambda_1^{3/2}, \\ -\int_0^\infty F_1^2\, F_1'\, dx &= -\,\frac{1}{3} F_1^3 \Big|_0^\infty = \frac{1}{3}\lambda_1^{3/2}, \end{aligned}$$

and finally integrating by parts

$$\begin{aligned} -2 \int_0^\infty \frac{\alpha}{x}\, F_1(x)\, F_1'\, dx &= -\int_0^\infty \frac{\alpha}{x}\, (F_1^2(x))'\, dx \\ &= -\,\frac{\alpha}{x} F_1^2(x) \Big|_0^\infty - \int_0^\infty \frac{\alpha}{x^2} F_1^2(x)\, dx \\ &= -\int_0^\infty \frac{\alpha}{x^2} F_1^2(x)\, dx \le 0. \end{aligned}$$

This finally implies

$$0 \le \int_0^\infty (V + 2F_1')^2\, dx = \int_0^\infty V^2\, dx - \frac{8}{3}\,\lambda_1^{3/2} - \int_0^\infty \frac{\alpha}{x^2} F_1^2(x)\, dx \tag{2.5}$$

$$\le \int_0^\infty V^2\, dx - \frac{8}{3}\,\lambda_1^{3/2}. \tag{2.6}$$

The ground state of the operator $\mathcal{H}_{\alpha+1} - 2F'$ equals

$$v_2 = (D_\alpha - F_1)u_2,$$

its eigenvalue equals λ_2 and we can apply similar computations elliminating it. This would lead us to the operator $H_{\alpha+2} - 2F_1' - 2F_2'$ whose ground state equals

$$v_3 = (D_{\alpha+1} - F_2)(D_\alpha - F_1)u_3,$$

etc.
The proof is complete.

3. Appendix

3.1. **Operator** $A(\alpha)$**.** In this section we justify equations (2.3) and (2.4). Let us start with computing the operator $A(\alpha)$

$$A(\alpha) = (D_\alpha^* - F_1)\,(D_\alpha - F_1) = \left(D_\alpha^* - \frac{D_\alpha u_1}{u_1}\right)\left(D_\alpha - \frac{D_\alpha u_1}{u_1}\right)$$

$$= D_\alpha^* D_\alpha - D_\alpha^* \frac{D_\alpha u_1}{u_1} - \frac{D_\alpha u_1}{u_1} D_\alpha + \left(\frac{D_\alpha u_1}{u_1}\right)^2 \tag{3.1}$$

Clearly

$$D_\alpha^* D_\alpha = \left(-\frac{d}{dx} - \frac{\alpha}{x}\right)\left(\frac{d}{dx} - \frac{\alpha}{x}\right) = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2}. \tag{3.2}$$

The second term in (3.1) equals

$$-D_\alpha^* \frac{D_\alpha u_1}{u_1} = -\left(-\frac{d}{dx} - \frac{\alpha}{x}\right)\left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right)$$

$$= \frac{u_1''}{u_1} - \left(\frac{u_1'}{u_1}\right)^2 + \frac{\alpha}{x^2} + \frac{u_1'}{u_1}\frac{d}{dx} - \frac{\alpha}{x}\frac{d}{dx} - \frac{\alpha^2}{x^2} + \frac{\alpha}{x}\frac{u_1'}{u_1}. \tag{3.3}$$

Similarly we have

$$-\frac{D_\alpha u_1}{u_1} D_\alpha = -\left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right)\left(\frac{d}{dx} - \frac{\alpha}{x}\right)$$

$$= -\frac{u_1'}{u_1}\frac{d}{dx} + \frac{\alpha}{x}\frac{d}{dx} + \frac{\alpha}{x}\frac{u_1'}{u_1} - \frac{\alpha^2}{x^2}, \tag{3.4}$$

and

$$\left(\frac{D_\alpha u_1}{u_1}\right)^2 = \left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right)^2 = \left(\frac{u_1'}{u_1}\right)^2 - 2\frac{\alpha}{x}\frac{u_1'}{u_1} + \frac{\alpha^2}{x^2}. \tag{3.5}$$

Adding together (3.2)-(3.5) we find

$$A(\alpha) = -\frac{d^2}{dx^2} + \frac{u_1''}{u_1}.$$

After using the equation

$$-u_1'' + \frac{\alpha(\alpha-1)}{x^2}u_1 - Vu_1 = -\lambda_1 u_1$$

we finally arrive at

$$A(\alpha) = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2} - V + \lambda_1$$

3.2. **Operator** $A_1(\alpha)$**.**

$$\begin{aligned} A_1(\alpha) = &\left(D_\alpha - \frac{D_\alpha u_1}{u_1}\right)\left(D_\alpha^* - \frac{D_\alpha u_1}{u_1}\right) \\ &= D_\alpha D_\alpha^* - D_\alpha \frac{D_\alpha u_1}{u_1} - \frac{D_\alpha u_1}{u_1} D_\alpha^* + \left(\frac{D_\alpha u_1}{u_1}\right)^2. \end{aligned} \tag{3.6}$$

Then

$$D_\alpha D_\alpha^* = \left(\frac{d}{dx} - \frac{\alpha}{x}\right)\left(-\frac{d}{dx} - \frac{\alpha}{x}\right) = -\frac{d^2}{dx^2} + \frac{\alpha(\alpha+1)}{x^2}. \tag{3.7}$$

$$\begin{aligned} - D_\alpha \frac{D_\alpha u_1}{u_1} = &-\left(\frac{d}{dx} - \frac{\alpha}{x}\right)\left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right) \\ &= -\frac{u_1''}{u_1} + \left(\frac{u_1}{u_1}\right)^2 - \frac{\alpha}{x^2} - \frac{u_1'}{u_1}\frac{d}{dx} + \frac{\alpha}{x}\frac{d}{dx} - \frac{\alpha^2}{x^2} + \frac{\alpha}{x}\frac{u_1'}{u_1}. \end{aligned} \tag{3.8}$$

Moreover,

$$\begin{aligned} - \frac{D_\alpha u_1}{u_1} D_\alpha^* = &-\left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right)\left(-\frac{d}{dx} - \frac{\alpha}{x}\right) \\ &= \frac{u_1'}{u_1}\frac{d}{dx} - \frac{\alpha}{x}\frac{d}{dx} + \frac{\alpha}{x}\frac{u_1'}{u_1} - \frac{\alpha^2}{x^2}, \end{aligned} \tag{3.9}$$

and as in (3.5)

$$\left(\frac{D_\alpha u_1}{u_1}\right)^2 = \left(\frac{u_1'}{u_1} - \frac{\alpha}{x}\right)^2 = \left(\frac{u_1'}{u_1}\right)^2 - 2\frac{\alpha}{x}\frac{u_1'}{u_1} + \frac{\alpha^2}{x^2}. \tag{3.10}$$

Adding together (3.7)-(3.10) we arrive at

$$\begin{aligned} A_1(\alpha) &= -\frac{d^2}{dx^2} - \frac{u_1''}{u_1} + 2\left(\frac{u_1'}{u_1}\right)^2 \\ &= -\frac{d^2}{dx^2} + \frac{u_1''}{u_1} - 2\left(\frac{u_1''}{u_1} - \left(\frac{u_1'}{u_1}\right)^2\right) \\ &= -\frac{d^2}{dx^2} + \frac{\alpha(\alpha-1)}{x^2} - V + \lambda_1 - 2\left(\frac{u_1'}{u_1} - \frac{\alpha}{x} + \frac{\alpha}{x}\right)' \\ &= -\frac{d^2}{dx^2} + \frac{\alpha(\alpha+1)}{x^2} - V + \lambda_1 - 2F_1' = \mathcal{H}_{\alpha+1} + \lambda_1 - 2F_1'. \end{aligned}$$

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ARI LAPTEV: IMPERIAL COLLEGE LONDON, 180 QUEEN'S GATE, LONDON SW7 2AZ, UK, AND ST PETERSBURG UNIVERSITY, RUSSIA

E-mail address: a.laptev@imperial.ac.uk