Many Particle Hardy Inequalities

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Abstract. In this paper we prove three different types of the so-called many-particle Hardy inequalities. One of them is a "classical type" which is valid in any dimension $d \neq 2$. The second type deals with two-dimensional magnetic Dirichlet forms where every particle is supplied with a solenoid. Finally we show that Hardy inequalities for Fermions hold true in all dimensions.

1. Introduction

Hardy inequalities play an important role in analysis. The classical one states that for $u \in H_0^1(0,\infty)$

$$\int_0^\infty \Big|\frac{du}{dx}\Big|^2 dx \geq \frac{1}{4}\int_0^\infty \frac{|u|^2}{|x|^2}dx. \tag{1.1}$$

The standard Hardy inequality (away from a point) for functions $u \in H^1(\mathbb{R}^d)$ reads for $d \geq 3$

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}dx. \tag{1.2}$$

There are many other inequalities which also are called Hardy inequalities, see for instance the survey paper by E.B. Davies [3] and the books of V.G. Maz'ya [13] and Kufner and Opic [7].

In the present paper we shall investigate a kind of Hardy inequalities which might be called many-particle Hardy inequalities. They can

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be related to some Schrödinger operators and have some interesting geometrical aspects.

Pick N a positive integer and consider N particles. This means we consider $x \in \mathbb{R}^{dN}$, where $x = (x_1, x_2, \dots, x_N)$ with $x_i = (x_{i,1}, x_{i,2} \dots, x_{i,d}) \in \mathbb{R}^d$. We define r_{ij} by

$$r_{ij} = |x_i - x_j| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2}.$$

We will write sometimes $\Delta_i = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_{i,k}^2}$ so that $\Delta = \sum_{i=1}^{N} \Delta_i$. Similarily we write sometimes ∇_i for the gradient associated to the i-th particle.

We have three groups of results. The first one deals with the "standard" Hardy inequality for many particles saying that

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_{x_j} u|^2 \, dx \geq \mathcal{C}(d, N) \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2} \, dx. \tag{1.3}$$

In Sections 4.1-4.3 we prove that this inequality holds for $d \geq 3$, $u \in H^1(\mathbb{R}^{dN})$ with a constant $\mathcal{C}(d, N)$, such that $c_1 N^{-1} \leq \mathcal{C}(d, N) \leq c_2 N^{-1}$, where $c_1, c_2 > 0$. The Hardy inequality (1.3) also holds for one-dimensional particles. In this case the function u is assumed to be equal to zero on diagonals $x_i = x_j$. We find in this case that $\mathcal{C}(1, N) = 1/2$ and that this constant is sharp.

In section 4.4 we consider the two-dimensional case and obtain a version of the Hardy inequality for magnetic multi-particle Dirichlet forms with Aharonov–Bohm type vector potentials attached to every particle. Let $x_j = (x_{j1}, x_{j2}) \in \mathbb{R}^2$, $j = 1, 2, \dots, N$, and let

$$\mathbf{F_j} = \alpha \Big(- \sum_{k \neq j} \frac{x_{j2} - x_{k2}}{r_{jk}^2}, \sum_{k \neq j} \frac{x_{j1} - x_{k1}}{r_{jk}^2} \Big), \tag{1.4}$$

where $\alpha \in \mathbb{R}$. Then we shall prove that

$$\int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F_j}) u|^2 \, dx \geq D_{N,\alpha} \int_{\mathbb{R}^{2N}} |u|^2 \Big(\sum_{k \neq j} \frac{1}{r_{kj}^2} \Big) \, dx. \tag{1.5}$$

The explicit value for the constant $D_{N,\alpha}$ depends on the "degree of rationality " of the magnetic flux α.

Our third result concerns the inequality (1.3) for fermions, i.e. the anti-symmetric functions in $H^1(\mathbb{R}^{dN})$. It turned out that in this case the Hardy inequality (1.3) holds true in all dimensions and if $d \geq 2$, then

$$\mathcal{C}(d, N) \geq \frac{d^2}{N}, \tag{1.6}$$

see Section 4.5.

2. Main results

2.1. Hardy inequalities for d-dimenional particles with $d \geq 3$.

Theorem 2.1. *Assume that $d \geq 3$, $N \geq 2$ and let $u \in H^1(\mathbb{R}^{dN})$. Let us define*

$$\mathcal{C}(d,N) = \inf_{u \in H^1(\mathbb{R}^{dN})} \frac{\int_{\mathbb{R}^{dN}} |\nabla u|^2 dx}{\sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2}\, dx}. \tag{2.1}$$

Then

$$\mathcal{C}(d,N) \geq (d-2)^2 \max\left\{ \frac{1}{N}, \frac{1}{1+\sqrt{1+\frac{3(d-2)^2}{2(d-1)^2}(N-1)(N-2)}} \right\}. \tag{2.2}$$

Remarks 2.2.

(i) *Hardy inequalities of this type cannot hold for general functions $u \in H^1(\mathbb{R}^{dN})$, $d = 1,2$.*

(ii) *For large values of N and $d \leq 6$ the maximum in* (2.2) *is given by the second term.*

(iii) *There is a very simple way of obtaining Hardy inequalities like above with a substantially weaker constant. Starting from* (1.2) *and noting that for any fixed $y \in \mathbb{R}^d$, $d \geq 3$,*

$$-\Delta \geq \frac{(d-2)^2}{4} \frac{1}{|x-y|^2},$$

we obtain

$$-\Delta_i - \Delta_j \geq \frac{(d-2)^2}{2} \frac{1}{r_{ij}^2}$$

in the quadratic form sense. Adding this up we would get

$$-\Delta \geq \frac{(d-2)^2}{2N-2} \sum_{i<j}^{N} \frac{1}{r_{ij}^2}$$

in the sense of quadratic forms and this is weaker than (2.2) *by a factor of more than two for large N and $d = 3$.*

(iv) *The bounds for $\mathcal{C}(d,N)$ are not sharp. Actually for the lower bound we use only the information from the derivation for the 3-particle case, i.e. $N = 3$. There is certainly a lot of room for improvement, though it is not clear how to get explicit better bounds. It is unclear what the optimal distribution of $\{x_j\}$ is as $N \to \infty$. Let $R(x,y,z)$ be the circumradius of the triangle* *with vertices x,y,z and suppose that the best asymptotic configuration of points could be described by a probability measure μ on $\mathbb{R}^d$. Let*

$$K = sup_\mu \frac{\int\int\int R^{-2}(x,y,z)\ d\mu(x)d\mu(y)d\mu(z)}{\int\int |x-y|^{-2}\ d\mu(x)d\mu(y)}.$$

Then applying (4.6)*, see below, one can obtain a much better estimate of the constant $C(d,N)$ for large N given by the inequality*

$$\lim_{N\to\infty} N\,\mathcal{C}(d,N) \geq \frac{(d-2)^2}{2+K}.$$

Note that the integral

$$C^2(\mu) = \int\int\int R^{-2}(x,y,z)\ d\mu(x)d\mu(y)d\mu(z)$$

is known as Menger-Melnikov curvature of the measure μ, see [12]*,* [15]*. Finding the value of K is an interesting open problem.*

The next theorem shows that the estimate $\mathcal{C}(d,N) = O(N^{-1})$, as $N \to \infty$, cannot be improved.

Theorem 2.3. *Let $\varphi \in H^1(\mathbb{R}^d)$, $d \geq 3$, and define*

$$M(\varphi) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^2} dxdy$$

and

$$\mathcal{D}(d) = \inf_{\varphi\in C_0^\infty(\mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} |\nabla\varphi|^2 dx \int_{\mathbb{R}^d} |\varphi|^2 dx}{M(\varphi)}.$$

Then

$$\mathcal{C}(d,N) \leq \frac{2\mathcal{D}(d)}{N-1}. \tag{2.3}$$

For numerical upper bounds see (4.15) and also Remark 4.2.

Corollary 2.4. *For any N and $d \geq 3$ there is a constant $\mathcal{C}' = \mathcal{C}'(d,N)$ such that the operator in $\mathbb{R}^{dN}$*

$$-\Delta - \mathcal{C}'(d,N) \sum_{1\leq i<j\leq N} \frac{1}{r_{ij}^2}$$

is not bounded from below and such that

$$\mathcal{C}'(d,N) \leq \frac{c(d)}{N}$$

with some $c(d) > 0$.

Corollary 2.4 could be obtained by explicit calculation, see (4.13) and (4.14). Indeed, it follows from Theorem 2.3 that there is a function, $\varphi \in H^1(\mathbb{R}^{dN})$ and a constant c such that

$$\int_{\mathbb{R}^{dN}} |\nabla\varphi|^2 - \frac{c}{N}\sum_{1\le i<j\le N}\int_{\mathbb{R}^{dN}} \frac{1}{r_{i,j}^2}|\varphi|^2 dx < 0. \tag{2.4}$$

Since both $-\Delta$ and the potential term show the same scaling, then if we replace $\varphi(x)$ by $\varphi(\lambda x)$ and normalize we can make the expression in (2.4) as negative as we want.

2.2. **Hardy inequality for 1D particles.**

Theorem 2.5. *Let*

$$\mathcal{N}_N = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \bigm| x_i = x_j \text{ for some } i \neq j\}.$$

Suppose that $u \in H_0^1(\mathbb{R}^N \setminus \mathcal{N}_N)$ *then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \sum_{i<j}^{N} \frac{1}{r_{ij}^2} dx. \tag{2.5}$$

The constant $1/2$ *is sharp.*

Remark 2.6. *One can get easily an inequality like* (2.5) *with an* N*-dependent constant instead of* $1/2$ *by using* (1.1). *First note that* (1.1) *can be rewritten such that for any* $y \in \mathbb{R}$ *and* $u \in H_0^1(\mathbb{R}\setminus\{y\})$

$$\int_{-\infty}^{\infty} \Big|\frac{du}{dx}\Big|^2 dx \ge \frac{1}{4}\int_{-\infty}^{\infty} \frac{|u|^2}{|x-y|^2} dx. \tag{2.6}$$

Now consider $N = 2$ *and note that* (2.6) *implies for* $u \in H_0^1(\mathbb{R}^2 \setminus \mathcal{N}_2)$

$$\int_{\mathbb{R}_2} \Big|\frac{\partial u}{\partial x_i}\Big|^2 dx \ge \frac{1}{4}\int_{\mathbb{R}^2} \frac{|u|^2}{|x_1 - x_2|^2} dx$$

for $i = 1, 2$ *so that adding up we get* $\|\nabla u\|^2 \ge \frac{1}{2}\|u/r_{12}\|^2$. *If we would continue like this we would get instead of the* $1/2$ *in* (2.5), $\frac{1}{2N-2}$ *as in* (iii) *of Remark 2.2, a much weaker bound tending to zero for* $N \to \infty$.

2.3. **Magnetic Hardy inequalities in 2D.**

Let the vector field $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_N)$ be defined by (1.4) and let

$$D_{N,\alpha} = \min_{l=1,\dots,N-1}\Big(\frac{\min_{k\in\mathbb{Z}}|k - l\alpha|}{l}\Big)^2. \tag{2.7}$$

Theorem 2.7. * *The following magnetic Hardy inequality for two-dimensional particles holds true*

$$\int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq D_{N,\alpha} \int_{\mathbb{R}^{2N}} |u|^2 \Big(\sum_{k \neq j} \frac{1}{r_{kj}^2} \Big) \, dx.$$

This inequality could be considered as a version of a 2D Hardy inequality by Laptev-Weidl [8] for Aharonov-Bohm magnetic Dirichlet forms and its generalisation obtained by A. Balinsky [1].

2.4. **Hardy inequalities for fermions.** Let us consider anti-symmetric functions $u(x)$, $x = (x_1, x_2, \dots, x_N)$, $x_j \in \mathbb{R}^d$, such that

$$u(x_1, \dots, x_i, \dots, x_j, \dots, x_N) = -u(x_1, \dots, x_j, \dots, x_i, \dots, x_N)$$

for all pairs (i, j), $i \neq j$.

Theorem 2.8. *For any $d = 1, 2, \dots,$ and anti-symmetric function $u \in H^1(\mathbb{R}^{dN})$ we have*

$$\sum_j \int_{\mathbb{R}^{dN}} |\nabla_{x_j} u|^2 \, dx \geq \frac{d^2}{N} \sum_{i<j} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2} \, dx. \tag{2.8}$$

Remark 2.9. *The latter inequality could be improved for large N. By using arguments from* [9] *and* [10] *we expect that for large N the N dependence of the constant in* (2.8) *could be improved to $N^{-1/3}$.*

It has recently been shown in [5] *that there is a constant C_d such that $\mathcal{C}(d, N) \geq C_d \, N^{-1/3}$.*

3. Some auxiliary results

In this section we consider several simple results of analytical and geometrical character and start with a simple but crucial inequality.

Lemma 3.1. *Let $u \in H^1(\mathbb{R}^m)$, $m \geq 1$ and let*

$$\mathcal{F} = \big(\mathcal{F}_1(x), \mathcal{F}_2(x), \dots, \mathcal{F}_m(x) \big)$$

be a vectorfield in $\mathcal{F} : \mathbb{R}^m \mapsto \mathbb{R}^m$ whose components and their first derivatives are uniformly bounded in $\mathbb{R}^m$. Then

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\Big(\int_{\mathbb{R}^m} |u|^2 \operatorname{div} \mathcal{F} dx \Big)^2}{\int_{\mathbb{R}^m} |u|^2 |\mathcal{F}|^2 dx}. \tag{3.1}$$

*The energy integral in the right hand side of (1.5) appears when studying the Fractional Quantum Hall Effect. It has been considered in [11], [6], where the fractional filling factor has been explained by attaching to each electron an infinitely thin magnetic solenoid carrying an Aharonov-Bohm flux (each electron bound to a flux tube has been called a "composite particle", [6]).

Proof.
We use the Cauchy-Schwarz inequality and partial integration. Indeed,

$$\Big|\int_{\mathbb{R}^m} |u|^2 \mathrm{div}\mathcal{F} dx\Big| = 2|\Re \int_{\mathbb{R}^m} \langle \mathcal{F},\, \nabla u\rangle \overline{u} dx| \\ \leq 2\Big(\int_{\mathbb{R}^m} |u|^2|\mathcal{F}|^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^m} |\nabla u|^2 dx\Big)^{1/2}.$$

Squaring this inequality completes the proof. □

The standard Hardy inequality (away from a point), (1.2) for $m \geq 3$ can be obtained by choosing

$$\mathcal{F}_\epsilon = \frac{x}{|x|^2+\epsilon^2}.$$

We pick $u \in H^1(\mathbb{R}^m)$ and insert $\mathcal{F}_\epsilon$ into (3.1) and obtain

$$\mathrm{div}\mathcal{F}_\epsilon = \frac{m}{|x|^2+\epsilon^2} - \frac{2|x|^2}{(|x|^2+\epsilon^2)^2} \geq \frac{m-2}{|x|^2+\epsilon^2}$$

so that

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{(m-2)^2}{4}\int_{\mathbb{R}^m} \frac{|u|^2}{|x|^2+\epsilon^2} dx.$$

$C_0^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and therefore $\epsilon \to 0$ gives (1.2).

The next lemma is related to the so-called Melnikov-Menger curvature and could be found, for example, in [15].

Lemma 3.2. *Define for three points $x_i, x_j, x_k \in \mathbb{R}^d$,*

$$b_{ijk} = \frac{\langle x_i - x_j,\ x_i - x_k\rangle}{r_{ij}^2 r_{ik}^2} + \frac{\langle x_j - x_i,\ x_j - x_k\rangle}{r_{ij}^2 r_{jk}^2} + \frac{\langle x_k - x_i,\ x_k - x_j\rangle}{r_{ik}^2 r_{jk}^2}.$$

Let R_{ijk} be the circumradius of the triangle with corners x_i, x_j, x_k. Then

$$b_{ijk} = \frac{1}{2R_{ijk}^2}, \quad \text{if} \quad d \geq 2 \qquad \text{and} \qquad b_{ijk} = 0, \quad \text{if} \quad d = 1.$$

Proof. Let $a = x_i - x_j$ and $b = x_i - x_k$. Then

$$b_{ijk} = \frac{(a\cdot b)}{|a|^2|b|^2} - \frac{a\cdot(b-a)}{|a|^2|b-a|^2} - \frac{b\cdot(a-b)}{|b|^2|b-a|^2} \\ = \frac{2\big(|a|^2|b|^2 - (a\cdot b)^2\big)}{|a|^2|b|^2|b-a|^2} = \frac{2\sin^2\phi}{r_{jk}^2}. \tag{3.2}$$

Here ϕ is the angle between a and b. The relation between the circumradius and the angle follws from the sine-theorem. Clearly if $x_i, x_j, x_k \in \mathbb{R}$ and not all of them equal, $R_{ijk} = \infty$. □

The next statements are concerned with two inequalities for triangles, see also [14].

Lemma 3.3. *Let R be the circumradius of a triangle with sides with side lenghts a, b, c then*

$$\frac{1}{R^2} \leq \frac{9}{a^2+b^2+c^2} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{3.3}$$

Both inequalities are equalities for the equilateral triangle.

Proof. This is an easy consequence of the sine-theorem and a Lagrange multiplyer argument. Indeed notice that $R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$ where the angles α, β, γ correspond to the angle at the corner opposite to the sides with side lengths a respectively b, c. We show the first inequality. This reads

$$4R^2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) \leq 9R^2. \tag{3.4}$$

It hence suffices to show that for $\alpha+\beta+\gamma = \pi$, $\sin^2\alpha+\sin^2\beta+\sin^2\gamma \leq 9/4$. So we look at

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \lambda(\alpha+\beta+\gamma-\pi) = \max!$$

Differentiation leads to

$$\sin 2\alpha + \lambda = 0,\ \sin 2\beta + \lambda = 0,\ \sin 2\gamma + \lambda = 0,\ \alpha+\beta+\gamma = \pi$$

and this implies that

$$\sin 2\alpha = \sin 2\beta = \sin 2\gamma,\ \ \alpha+\beta+\gamma = \pi.$$

There are three solutions, namely $\alpha = \beta = \gamma$, $\alpha = \beta = \pi/2$, $\gamma = 0$ and finally $\alpha = \pi$, $\beta = \gamma = 0$. If we insert the values into (3.4) we get the desired result.

For the other inequality we have to show that

$$P = \Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big)(a^2+b^2+c^2) \geq 9.$$

and this can be seen by multiplication which yields

$$P = 3 + a^2/b^2 + b^2/a^2 + a^2/c^2 + c^2/a^2 + b^2/c^2 + c^2/b^2 \geq 9$$

since $a^2/b^2 + b^2/a^2 \geq 2$ and similarily for the other fractions above.

□

The following two statements can be checked by straight forward computations.

Lemma 3.4. *Let $x_j \in \mathbb{R}^d$, $j = 1, 2, 3$. Then*

$$r_{12}^2 + r_{13}^2 + r_{23}^2$$
$$= 2\Big(\langle x_1 - x_2, x_1 - x_3\rangle + \langle x_2 - x_1, x_2 - x_3\rangle + \langle x_3 - x_1, x_3 - x_2\rangle\Big).$$

Lemma 3.5. *Let $x_j \in \mathbb{R}^d$, $j = 1, 2, \ldots, N$. Then*

$$N \sum_{j=1}^{N} \Delta_{x_j} = \sum_{1 \le j < k \le N} (\nabla_{x_j} - \nabla_{x_k})^2 + \Big(\sum_{j=1}^{N} \nabla_{x_j}\Big)^2.$$

Finally we need a statement which could be considered as two versions of Hardy's inequalities for three particles.

Lemma 3.6. *Let $x_1, x_2, x_3 \in \mathbb{R}^d$, $d \ge 2$, and let*

$$\rho^2 = r_{12}^2 + r_{13}^2 + r_{23}^2.$$

Then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 dx \ge 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{\rho^2} dx. \tag{3.5}$$

Furthermore if $R(x)$ is the circumradius of the triangle with vertices x_1, x_2, x_3, then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 dx \ge \frac{(d-1)^2}{3} \int_{\mathbb{R}^{3d}} \frac{|u|^2}{R^2} dx. \tag{3.6}$$

Proof. This follows from a simple direct calculation. Let $\mathcal{F} = \mathcal{G}$ in (3.1), where

$$\mathcal{G} = \frac{1}{\rho^2}\Big(2x_1 - x_2 - x_3,\ 2x_2 - x_1 - x_3,\ 2x_3 - x_1 - x_2\Big). \tag{3.7}$$

Then by applying Lemma 3.1 we easily work out by using the identity given in Lemma 3.4, that

$$\operatorname{div} \mathcal{G} = \frac{6(d-1)}{\rho^2}$$

and

$$|\mathcal{G}|^2 = \frac{3}{\rho^2}.$$

We insert these equalities into (3.1) and obtain (3.5). To be more precise we first consider $\mathcal{G}_\epsilon$ where the denominator in (3.7) is replaced by $\rho^2 + \epsilon^2$. Then as in the proof of the standard Hardy-inequality the result follows as ϵ tends to zero. Finally in order to prove (3.6) we use the inequality from Lemma 3.3, which tells us that

$$\rho^2 \le 9R^2.$$

Hence (3.6) follows immediately from (3.5). □

Remarks 3.7.

(i) *For one-dimensional particles the circumradius is equal to infinity and therefore* (3.6) *becomes trivial.*

(ii) *However, we do not believe that the constant in* (3.6) *is sharp. Perhaps one can find a suitable $\mathcal{F}$ so that one can directly obtain a Hardy-type inequality for R^{-2}.*

4. Proofs of main results.

4.1. **Proof of Theorem 2.1.**

A. Let us first give a simple proof of the inequality (2.1) which states that $\mathcal{C}(d,N) \geq (d-2)^2/N$.

For a function $u \in H^1(\mathbb{R}^{dN})$ we consider a vector field

$$\mathcal{F}_1(x_j, x_k) = (x_j - x_k) r_{jk}^{-2}, \qquad 1 \leq j < k \leq N.$$

Then by using arguments from the proof of Lemma 3.1 with the vector field $\mathcal{F}_1$ we find

$$\int_{\mathbb{R}^{dN}} |(\nabla_{x_j} - \nabla_{x_k}) u|^2 \, dx_j dx_k \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^{dN}} |u|^2 \left((\mathrm{div}_{x_j} - \mathrm{div}_{x_k}) \, \mathcal{F}_{\mathbf{1}} \right) dx \right)^2}{\int_{\mathbb{R}^{dN}} |u|^2 |\mathcal{F}_{\mathbf{1}}|^2 dx} = (d-2)^2 \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{jk}^2} \, dx_j dx_k. \tag{4.1}$$

Moreover, if we introduce the vector field

$$\mathcal{F}_2(x) = \frac{\sum_{j=1}^N x_j}{\left| \sum_{j=1}^N x_j \right|^2},$$

then using Lemma 3.1 with $\mathcal{F}_2$ we obtain

$$\int_{\mathbb{R}^{dN}} \Big| \sum_{j=1}^N \nabla_{x_j} u \Big|^2 \geq \frac{(d-2)^2 N^2}{4} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{\left| \sum_{j=1}^N x_j \right|^2} \, dx. \tag{4.2}$$

Adding the inequalities (4.1) and (4.2) up and using Lemma 3.5 we arrive at

$$\int |\nabla u|^2 dx \geq \frac{(d-2)^2}{N} \int_{\mathbb{R}^{dN}} \sum_{j<k} \frac{|u|^2}{r_{jk}^2} \, dx + \frac{(d-2)^2 N^2}{4N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{\left| \sum_{j=1}^N x_j \right|^2} \, dx. \tag{4.3}$$

The latter inequality implies the inequality $\mathcal{C}(d,N) \geq (d-2)^2/N$ and also gives a positive remainder term which is of order $O(N)$.

B. Let us now define

$$\mathfrak{F}_3 = (F_1, \dots, F_N), \tag{4.4}$$

where the F_j are given by

$$F_j = \sum_{k \neq j}^{N} \frac{x_j - x_k}{r_{jk}^2}. \tag{4.5}$$

In order to prove Theorem 2.1 we apply Lemma 3.1 for the vector field $\mathfrak{F}_3$ which is conveniently written as a vector with N elements which themselves are vectors with d entries. The divergence of $\mathfrak{F}_3$ can be similarily defined as

$$\operatorname{div} \mathfrak{F}_3 = \sum_{i=1}^{N} \nabla_i \cdot F_i$$

where $\nabla_i \cdot F_i = \operatorname{div}\, F_i$ and where the divergence is now with respect to a d-dimensional vector field.

Proposition 4.1. *Assume that $d \geq 3$ and $N \geq 2$. Let for an arbitrary $u \in H^1(\mathbb{R}^{dN})$*

$$T(d,N) = \int_{\mathbb{R}^{dN}} |\nabla u|^2 dx,$$

$$X(d,N) = \sum_{1 \leq 1 < j \leq N} \int_{\mathbb{R}^{dN}} \frac{1}{r_{ij}^2} |u|^2 dx$$

and

$$Z(d,N) = \sum_{1 \leq i < j < k \leq N} \int_{\mathbb{R}^{dN}} \frac{1}{R_{ijk}^2} |u|^2 dx,$$

where R_{ijk} is as in Lemma 3.2. Then

$$T(d,N) \geq (d-2)^2 \, \frac{X(d,N)^2}{2X(d,N) + Z(d,N)}. \tag{4.6}$$

Proof.
The proof is an easy calculation. We just note that

$$\operatorname{div}\, \mathfrak{F}_3 = 2(d-2) \sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}^2} \tag{4.7}$$

and that

$$|\mathfrak{F}_3|^2 = 2 \sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}^2} + \sum_{1 \leq i < j < k \leq N} \frac{1}{R_{ijk}^2}, \tag{4.8}$$

where we used Lemma 3.2. We just have to insert these expressions into (3.1) to obtain (4.6) proving the proposition. □

Consider now inequality (4.6). There are two possibilities to obtain from this quadratic inequality a linear inequality

a. First we can try to find an estimate such that

$$Z(d,N) \le k(d,N)X(d,N)$$

and this leads to

$$\mathcal{C}(d,N) \ge \frac{(d-2)^2}{2+k(d,N)}. \tag{4.9}$$

b. The other possibility is to find an estimate of the form

$$Z(d,N) \le \ell(d,N)T(d,N).$$

Indeed, with this estimate we get

$$T(d,N) \ge (d-2)^2 \frac{X(d,N)^2}{2X(d,N)+\ell(d,N)T(d,N)}$$

and this leads to the quadratic inequality

$$X(d,N)^2 - \frac{2X(d,N)T(d,N)}{(d-2)^2} - \frac{\ell(d,N)T(d,N)^2}{(d-2)^2} \le 0. \tag{4.10}$$

Therefrom we get by solving the corresponding quadratic equation

$$C(d,N) \ge \frac{(d-2)^2}{1+\sqrt{1+\ell(d,N)^2(d-2)^2}}. \tag{4.11}$$

case a.
We show that

$$\mathcal{C}(d,N) \ge \frac{(d-2)^2}{N}. \tag{4.12}$$

This is an easy consequence of the inequality (3.3) in Lemma 3.3. Indeed we just have to show (4.9) that $k(d,N) \le N-2$ and this can be seen by counting. Clearly $Z(d,N)$ consists of $\binom{N}{3}$ and $X(d,N)$ of $\binom{N}{2}$ terms. Finally we group each three particle coordinates together and apply Lemma 3.3. This gives (4.12) and an alternative proof of the result obtained in subsection **A**.

case b.
This case is more involved. We begin with considering three particles.

Note that for three d-dimensional particles with $d \ge 3$, (3.6) implies

$$\ell(d,3) \le \frac{3}{(d-1)^2}$$

so that we obtain for $N=3$ in (4.11)

$$T(d,3) \ge \frac{(d-2)^2}{1+\sqrt{1+\frac{3(d-2)^2}{d-1)^2}}} X(d,3).$$

We continue with the N-particle case and get by counting from (4.6) that

$$T(d,N) \ge \frac{2(d-1)^2}{3(N-1)(N-2)} Z(d,N).$$

From the quadratic inequality (4.10) we now infer that

$$T(d,N) \geq (d-2)^2 \frac{1}{1+\sqrt{1+\frac{3(d-2)^2}{2(d-1)^2}\,(N-1)(N-2)}} X(d,N).$$

This inequality together with (4.12) proves (2.2) and therefore the second part of Theorem 2.1. □

4.2. **Proof of Theorem 2.3.** Let $\varphi(x_i) \in H^1(\mathbb{R}^d)$ and consider for fixed N

$$u(x) = u_N(x) = \prod_{i=1}^{N} \varphi(x_i), \;\; x_i \in \mathbb{R}^d.$$

We observe that

$$\int_{\mathbb{R}^{dN}} |\nabla u_N(x)|^2 dx = N \int_{\mathbb{R}^d} |\nabla_1 \varphi(x)|^2 dx \left(\int_{\mathbb{R}^d} |\varphi(x)|^2 dx \right)^{N-1}. \tag{4.13}$$

Next we calculate

$$\sum_{1\leq i<j\leq N} \int_{\mathbb{R}^{dN}} \frac{1}{r_{ij}^2} |u_N|^2 dx = \tag{4.14}$$

$$\frac{N(N-1)}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi(x)|^2 \frac{1}{|x-y|^2} |\varphi(y)|^2 dxdy \left(\int_{\mathbb{R}^d} |\varphi(x)|^2 dx \right)^{N-2}.$$

Substituting the expressions from (4.13) and (4.14) into (2.1) we complete the proof. □

Here we provide a numerical value for the right hand side in (2.3) and therefore an estimate from above for the constant $\mathcal{C}(d,N)$.

Let us choose

$$\varphi(x) = e^{-|x|^2/2}.$$

Then

$$\int_{\mathbb{R}^d} |\varphi(x)|^2\, dx = \frac{1}{2}\, |\mathbb{S}^{d-1}|\, \Gamma(d/2), \qquad \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2\, dx = \frac{d}{4}\, |\mathbb{S}^{d-1}|\, \Gamma(d/2).$$

Straight forward computations give us

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^2\, |\varphi(y)|^2}{|x-y|^2}\, dxdy = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Substituting all the expressions into (2.3) we obtain

$$\mathcal{C}(d,N) \leq \frac{2d}{2(N-1)}\, \pi^{d/2}\, \Gamma(d/2). \tag{4.15}$$

In particular,

$$\mathcal{C}(3,N) \leq \frac{1}{N-1}\, \frac{3\pi^2}{4},$$

i.e. $0.43 < \mathcal{C}(3,N) < 3.69$. For the three particle system using the estimate from below provided by Theorem 2.1 we have

$$\frac{1}{1+\sqrt{7}/2} \le \mathcal{C}(3,3) \le \frac{3\pi^2}{8}.$$

Remark 4.2. *It follows from* (2.2) *that the gap between the lower and upper bounds obtained in Theorem 2.1 and in formula* (4.15) *is growing with respect to d.*

4.3. **Proof of Theorem 2.5.** The inequality (2.5) follows immediately from Lemma 3.1 with $\mathcal{F}$ defined by (4.4), (4.5) and the relations (4.7) and (4.8). It only remains to observe that by Lemma 3.2 the second sum in (4.8) is equal to zero for $d = 1$.

Let us now prove that the constant 1/2 appearing in (2.5) is sharp. It is enough to show that for any $\varepsilon > 0$ there is a function $v = v_\varepsilon$ such that

$$\int_{\mathbb{R}^N} |\nabla v(x)|^2\, dx \le \Big(\frac{1}{2} + \varepsilon\Big) \int_{\mathbb{R}^N} |v(x)|^2 \sum_{i<j}^{N} \frac{1}{r_{ij}^2}\, dx. \tag{4.16}$$

Let $\alpha = 1/4 + \delta$

$$v(x) = \Pi_{i\neq j}(x_i - x_j)^{2\alpha} e^{-|x|}.$$

Then

$$\partial_{x_i} v = 2\alpha v \sum_{j:j\neq i} \frac{1}{x_i - x_j} - v\, \frac{x_i}{|x|}.$$

Therefore

$$\begin{aligned}
(4.17)\quad |\nabla v|^2 &= \sum_{i=1}^{N} |\partial_{x_i} v|^2 \\
&= \sum_{i=1}^{N} \Big(4\alpha^2 v^2 \Big(\sum_{j:j\neq i} \frac{1}{(x_i-x_j)^2} + \sum_{j,k:j,k\neq i,\, j\neq k} \frac{1}{x_i - x_j}\,\frac{1}{x_i - x_k}\Big) \\
&\qquad - 4\alpha v^2 \sum_{j:j\neq i} \frac{1}{x_i - x_j}\,\frac{x_i}{|x|} + v^2\, \frac{x_i^2}{|x|^2}\Big).
\end{aligned}$$

Note that by Lemma 3.2

$$\sum_{i=1}^{N} \sum_{j,k:j,k\neq i,\, j\neq k} \frac{1}{x_i - x_j}\,\frac{1}{x_i - x_k} = \sum_{i=1}^{N} \sum_{j,k:j,k\neq i,\, j\neq k} \frac{x_i - x_j}{r_{ij}^2}\,\frac{x_i - x_k}{r_{ik}^2} = 0.$$

Moreover the identity

$$\sum_{j:j\neq i} \frac{1}{x_i - x_j}\,\frac{x_i}{|x|} = \frac{1}{|x|} \sum_{j:j\neq i} \Big(1 - \frac{x_j}{x_j - x_i}\Big)$$

implies

$$\sum_{i=1}^{N}\sum_{j:j\neq i}\frac{1}{x_i-x_j}\,\frac{x_i}{|x|}=\frac{1}{2|x|}\,N(N-1)\geq 0.$$

Therefore we obtain from (4.17)

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v|^2\,dx=\int_{\mathbb{R}^N}\Big(\sum_{i<j}\frac{8\alpha^2\,v^2}{r_{ij}^2}-\frac{2\alpha v^2}{|x|}\,N(N-1)+v^2\Big)\,dx\\ \leq 8\alpha^2\int_{\mathbb{R}^N}\sum_{i<j}\frac{v^2}{r_{ij}^2}\,dx\,\Big(1+\beta(\delta)\Big),\end{aligned}$$

where

$$\beta(\delta)=\frac{\int_{\mathbb{R}^N}v^2\,dx}{8(1/4+\delta)^2\int_{\mathbb{R}^N}v^2\sum_{i<j}\frac{1}{r_{ij}^2}\,dx}\to 0\quad\text{as}\quad\delta\to 0.$$

We conclude the proof by choosing δ small enough so that it satisfies the inequality $1/2+\varepsilon\geq 8(1/4+\delta)^2(1+\beta(\delta))$. □

4.4. **Proof of Theorem 2.7.** We begin with recalling two results obtained in the papers of [8] and [1] concerning the Hardy inequalities for Aharonov-Bohm magnetic Dirichlet forms.

a. One particle inequality. Let $x=(x_1,x_2)\in\mathbb{R}^2$, $\alpha\in\mathbb{R}$ and let $\mathbf{F}$ be the Aharonov-Bohm vector potential

$$\mathbf{F}=(F_1,F_2)=\alpha\Big(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\Big).$$

Lemma 4.3.

$$\int_{\mathbb{R}^2}|(i\nabla+\mathbf{F})u|^2\,dx\geq\min_{k\in\mathbb{Z}}(k-\alpha)^2\int_{\mathbb{R}^2}\frac{|u|^2}{|x|^2}\,dx.$$

Proof. Indeed, using polar coordinates (r,θ) we have $u(x)=\frac{1}{\sqrt{2\pi}}\sum_k u_k(r)e^{ik\theta}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^2}|(i\nabla+\mathbf{F})u|^2\,dx=\int_0^\infty\int_0^{2\pi}\Big(|u_r'|^2+\Big|\frac{iu_\theta'+\alpha u}{r}\Big|^2\Big)\,r\,d\theta\,dr\\ \geq\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\Big|\sum_k\frac{\alpha-k}{r}\,u_ke^{ik\theta}\Big|^2\,r\,d\theta\,dr=\int_0^\infty\sum_k\Big|\frac{\alpha-k}{r}\,u_k\Big|^2\,r\,d\theta\,dr\\ \geq\min_{k\in\mathbb{Z}}(k-\alpha)^2\int_{\mathbb{R}^2}\frac{|u|^2}{|x|^2}\,dx.\end{aligned}$$

□

b. Magnetic potentials with multiple singularities.

Assume that $\{z_1, z_2, \dots, z_n\}$ are n fixed different points in $\mathbb{C}$, $z_j = x_j + iy_j$ and $\alpha_j \in \mathbb{R}$. Let $\mathbf{F}$ the following vector potential

$$\mathbf{F} = \sum_{j=1}^{n} \alpha_j \Big(\frac{-y + y_j}{|z - z_j|^2}, \frac{x - x_j}{|z - z_j|^2} \Big), \quad z = x + iy.$$

This corresponds to Aharonov-Bohm magnetic vector fields placed in n points z_j with magnetic fluxes α_j. Let now $\Phi : \mathbb{C} \to \mathbb{C}$ be an analytic function with zero set $\{z_1, z_2, \dots, z_n\}$ and such that $\Phi(\infty) = \infty$.

Let $\{\xi_1, \xi_2, \dots, \xi_m\}$ be the zero set of Φ'_z and let $\{0, |\Phi(\xi_1)|, \dots, |\Phi(\xi_m)|\}$ be such that $0 \geq |\Phi(\xi_1)| \geq \dots \geq |\Phi(\xi_m)|$. Denote by $\mathcal{A}$ the pre-image of these points under the map $|\Phi| : \mathbb{C} \to \mathbb{R}_+$. For an arbitrary point $z \notin \mathcal{A}$ we define a curve γ_z obtained by $|\Phi|^{-1}(|\Phi(z)|)$. Let $\Omega_z \subset \mathbb{C}$ be a bounded domain defined by γ_z. We now consider a piecewise constant function

$$C_\Phi(z) = \Big(\frac{\min_{k \in \mathbb{Z}} |k - \sum_{j:\, z_j \in \Omega_z} \alpha_j|}{\sum_{j:\, z_j \in \Omega_z} 1} \Big)^2. \tag{4.18}$$

Lemma 4.4. (A. Balinsky) *The following Hardy inequality holds true*

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{F})u|^2 \, dxdy \geq \int_{\mathbb{R}^2} C_\Phi(z) \Big| \frac{\Phi'_z}{\Phi} \Big|^2 |u|^2 \, dxdy.$$

For the proof see [1].

c. Multi-particle case. Let now $z = (z_1, \dots, z_N)$, $z_j = x_{j1} + ix_{j2}$ and let $\Phi_j(z) = \Pi_{k \neq j}(z_j - z_k)$, $j, k = 1, \dots, N$. Then according Balinsky's lemma there are piecewise constants functions $C_{\Phi_j}(x)$ defined by (4.18), such that

$$\int_{\mathbb{R}^{2N}} |i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq \int_{\mathbb{R}^{2N}} C_{\Phi_j}(x) \Big| \frac{(\Phi_j)'_{z_j}(z)}{\Phi_j(z)} \Big|^2 |u|^2 \, dx.$$

A simple computation shows

$$\Big| \frac{(\Phi_j)'_{z_j}(z)}{\Phi_j(z)} \Big|^2 = \Big| \sum_{k \neq j} \frac{1}{z_j - z_k} \Big|^2 = \sum_{k,l \neq j} \frac{(x_j - x_k) \cdot (x_j - x_l)}{r_{jk}^2 r_{jl}^2}.$$

Note that $C_{\Phi_j}(x) \geq D_{N,\alpha}$, where $D_{N,\alpha}$ is defined by (2.7). Therefore we obtain

$$\int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq D_{N,\alpha} \int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} \Big| \sum_{k \neq j}^{N} \frac{1}{z_j - z_k} \Big|^2 |u|^2 \, dx$$

$$= D_{N,\alpha} \int \Big(\sum_{k \neq j}^{N} \frac{1}{r_{jk}^2} + \sum_{l \neq k, l, k \neq j}^{N} \frac{1}{R_{jkl}^2} \Big) |u|^2 \, dx.$$

We complete the proof by noticing that

$$\min_{x\in\mathbb{R}^{2N}} \sum_{l\neq k,l,k\neq j}^{N} R_{jkl}^{-2} = 0.$$

□

4.5. **Proof of Theorem 2.8.** Let us begin with a simple observation concerning odd functions in $\mathbb{R}^d$ which has been pointed out already in the classical paper of M.S. Birman [2].

Proposition 4.5. *Let $u(x) = -u(-x) \in H^1(\mathbb{R}^d)$, $d \geq 2$. Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \frac{d^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx.$$

Proof.
Let us introduce spherical coordinates $x = (r, \theta)$. Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} \Big(|u_r'|^2 + \frac{|\nabla_\theta u|^2}{r^2} \Big) \, r^{d-1} \, d\theta dr.$$

By using the 1-dimensional Hardy inequality with weight (see for example [7]) we obtain

$$\int_0^\infty \int_{\mathbb{S}^{d-1}} |u_r'|^2 r^{d-1} \, d\theta dr \geq \frac{(d-2)^2}{4} \int_0^\infty \int_{\mathbb{S}^{d-1}} |u|^2 r^{d-3} \, d\theta dr$$
$$= \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx.$$

It only remains to note that since u is an odd function it is orthogonal to constants on $\mathbb{S}^{d-1}$ and therefore

$$\int_0^\infty \int_{\mathbb{S}^{d-1}} |\nabla_\theta u|^2 \, r^{d-3} \, d\theta dr \geq (d-1) \int_0^\infty \int_{\mathbb{S}^{d-1}} |u|^2 \, r^{d-3} \, d\theta dr$$
$$= (d-1) \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx,$$

where $d-1$ is the second eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. □

We now consider an anti-symmetric function of two variables $x, y \in \mathbb{R}^d$.

Lemma 4.6. *For any anti-symmetric function $u(x,y) = -u(y,x) \in H^1(\mathbb{R}^{2d})$ we have*

$$\int_{\mathbb{R}^{2d}} |(\nabla_x - \nabla_y) u(x,y)|^2 dx dy \geq d^2 \int_{\mathbb{R}^4} \frac{|u(x,y)|^2}{|x-y|^2} dx dy.$$

Proof. We make an orthogonal coordinate transformation

$$s = \frac{1}{\sqrt{2}}(x+y), \;\; t = \frac{1}{\sqrt{2}}(x-y).$$

Thus $|x|^2 + |y|^2 = |s|^2 + |t|^2$ and

$$\nabla_s = \frac{1}{\sqrt{2}}(\nabla_x + \nabla_y), \;\; \nabla_t = \frac{1}{\sqrt{2}}(\nabla_x - \nabla_y), \;\; \Delta = \Delta_x + \Delta_y = \Delta_s + \Delta_t.$$

If we define the function $\tilde{u}(s,t)$ as

$$\tilde{u}(s,t) = u(x,y) = u\Big(\frac{s+t}{\sqrt{2}}, \frac{s-t}{\sqrt{2}}\Big),$$

then it is odd with respect to t, $\tilde{u}(s,-t) = -\tilde{u}(s,t)$. By using Proposition 4.5 we obtain

$$\int_{\mathbb{R}^{2d}} |\nabla_t \tilde{u}(s,t)|^2 \, dsdt \geq \frac{d^2}{4} \int_{\mathbb{R}^{2d}} \frac{|\tilde{u}|^2}{|t|^2} dsdt.$$

Transforming back to u and noting that $|t|^{-2} = 2|x-y|^{-2}$ we complete the proof. □

Let us note (cf. Lemma 3.5) that for $\xi \in \mathbb{R}^{dN}$

$$\sum_{j=1}^{N} |\xi_j|^2 = \frac{1}{N} \sum_{j<k} |\xi_j - \xi_k|^2 + \frac{1}{N} \Big| \sum_{j=1}^{N} \xi_j \Big|^2.$$

If $\hat{u}$ is the Fourier transform of the function u, then by using Lemma 4.6 we find

$$\begin{aligned}
\sum_j \int_{\mathbb{R}^{dN}} |\nabla_{x_j} u|^2 \, dx &= \sum_j \int_{\mathbb{R}^{dN}} |\xi_j \hat{u}|^2 \, d\xi \\
&= \frac{1}{N} \sum_{j<k} \int |(\xi_j - \xi_k)\hat{u}|^2 \, d\xi + \frac{1}{N} \int |\sum_j \xi_j \, \hat{u}|^2 \, d\xi \\
&= \frac{1}{N} \sum_{j<k} \int |(\nabla_{x_j} - \nabla x_k)\, u|^2 \, dx + \frac{1}{N} \int \Big| \sum_{j=1}^{N} \nabla_{x_j} u \Big|^2 \, dx \\
&\geq \frac{d^2}{N} \sum_{i<j} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2} \, dx.
\end{aligned}$$

In the latter inequality we can neglect the second integral and this completes the proof of Theorem 2.8. □

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