NEW BOUNDS ON THE LIEB-THIRRING CONSTANTS

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ABSTRACT. Improved estimates on the constants $L_{\gamma,d}$, for $1/2 < \gamma < 3/2$, $d \in \mathbb{N}$, in the inequalities for the eigenvalue moments of Schrödinger operators are established.

1. INTRODUCTION

Let us consider a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$-\Delta + V, \tag{1.1}$$

where V is a real-valued function. The inequalities

$$\operatorname{tr}(-\Delta + V)_-^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\, dx\,, \tag{1.2}$$

are known as Lieb-Thirring bounds and hold true with finite constants $L_{\gamma,d}$ if and only if $\gamma \ge 1/2$ for $d = 1$, $\gamma > 0$ for $d = 2$ and $\gamma \ge 0$ for $d \ge 3$. Here and in the following, $A_\pm = (|A| \pm A)/2$ denote the positive and negative parts of a self-adjoint operator A. The case $\gamma > (1 - d/2)_+$ was shown by Lieb and Thirring in [21]. The critical case $\gamma = 0$, $d \ge 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [8, 19, 22] and also [18, 7]. The remaining case $\gamma = 1/2$, $d = 1$ was verified in [25].

It is known that as soon as $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ and the constant $L_{\gamma,d}$ is finite, then we have Weyl's asymptotic formula

$$\begin{aligned}\lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \operatorname{tr}(-\Delta + \alpha V)_-^\gamma &= \lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 + \alpha V)_-^\gamma \frac{dx\,d\xi}{(2\pi)^d} \\ &= L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\, dx\,,\end{aligned} \tag{1.3}$$

where the so-called classical constant $L^{cl}_{\gamma,d}$ is defined by

$$L^{cl}_{\gamma,d} = (2\pi)^{-d} \int_{\mathbb{R}^d} (|\xi|^2 - 1)_-^\gamma \, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + \frac{d}{2} + 1)}\,, \quad \gamma \ge 0\,. \tag{1.4}$$

This immediately implies $L^{cl}_{\gamma,d} \le L_{\gamma,d}$.

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Until recently the sharp values of $L_{\gamma,d}$ were known only for $\gamma \geq 3/2$, $d = 1$, (see [21, 1]), where they coincide with $L^{cl}_{\gamma,d}$. In [17] Laptev and Weidl extended this result to all dimensions. They proved that $L_{\gamma,d} = L^{cl}_{\gamma,d}$, for $\gamma \geq 3/2$, $d \in \mathbb{N}$. Recently, Hundertmark, Lieb and Thomas showed in [15] that the sharp value of $L_{1/2,1}$ is equal to $1/2$.

The purpose of this paper is to give some new bounds on the constants $L_{\gamma,d}$ for $1/2 < \gamma < 3/2$ and all $d \in \mathbb{N}$ (see §4). In particular, one of our main results given in Theorem 4.1, says that

$$L_{\gamma,d} \leq 2L^{cl}_{\gamma,d}, \qquad 1 \leq \gamma < 3/2, \quad d \in \mathbb{N}, \tag{1.5}$$

whereas for large dimensions it was only known that $L_{\gamma,d} \leq C\sqrt{d}\, L^{cl}_{\gamma,d}$ with some constant $C > 0$.

For the important case $\gamma = 1$, $d = 3$ we have $L_{1,3} \leq 2L^{cl}_{1,3} < 0.013509$ compared with $L_{1,3} < 5.96677 L^{cl}_{1,3} < 0.040303$ obtained in [20] and its improvement $L_{1,3} < 5.21803 L^{cl}_{1,3} < 0.035246$ obtained in [5].

Note also that our estimates on the constant $L_{\gamma,d}$ imply that $L_{1,d} \leq 2L^{cl}_{1,d} < L^{cl}_{0,d}$ as was conjectured in [23].

In order to get our results we give a version of the proof obtained in [15] for matrix-valued potentials (see §3). Note that E.H.Lieb has informed us that the original proof obtained in [15] also works for matrix-valued potentials. After that in §4 we apply the equality $L_{\gamma,d} = L^{cl}_{\gamma,d}$, for $\gamma \geq 3/2$ and $d \in \mathbb{N}$ shown in [17] by using the "lifting" argument with respect to the dimension d suggested in [16]. The same arguments as in [17] yield the corresponding inequalities for Schrödinger operators with magnetic fields.

Finally, we are very grateful to L.E.Thomas who was also involved in the new proof of Theorem 3.1 as well as making many valuable remarks.

2. Notation and Auxiliary Material

Let $\mathbf{G}$ be a separable Hilbert space with the norm $\|\cdot\|_{\mathbf{G}}$ and the scalar product $\langle\cdot,\cdot\rangle_{\mathbf{G}}$ and let $0_{\mathbf{G}}$ and $1_{\mathbf{G}}$ be the zero and the identity operator on $\mathbf{G}$. Denote by $\mathcal{B}(\mathbf{G})$ the Banach space of all bounded operators on $\mathbf{G}$ and by $\mathcal{K}(\mathbf{G})$ the (separable) ideal of all compact operators. Let $\mathcal{S}_1(\mathbf{G})$ and $\mathcal{S}_2(\mathbf{G})$ be the classes of trace and Hilbert-Schmidt operators on $\mathbf{G}$ respectively. For a nonnegative operator $A \in \mathcal{K}(\mathbf{G})$

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq 0$$

is the ordered sequence of its eigenvalues (including multiplicities). We use the symbol "tr" to denote traces of operators (matrices) in different Hilbert spaces.

The Hilbert space $\mathbf{H} = L^2(\mathbb{R}^d, \mathbf{G})$ is the space of all measurable functions $u : \mathbb{R}^d \to \mathbf{G}$ such that

$$\|u\|^2_{\mathbf{H}} := \int_{\mathbb{R}^d} \|u\|^2_{\mathbf{G}} \, dx < \infty.$$

The Sobolev space $H^1(\mathbb{R}^d, \mathbf{G})$ consists of all functions $u \in \mathbf{H}$ whose norm

$$\|u\|^2_{H^1(\mathbb{R}^d,\mathbf{G})} = \sum_{k=1}^{d} \|\partial u/\partial x_k\|^2_{\mathbf{H}} + \|u\|^2_{\mathbf{H}}$$

is finite. Obviously the quadratic form

$$h[u,u] = \sum_{k=1}^{d} \|\partial u/\partial x_k\|^2_{\mathbf{H}}$$

is closed in $L^2(\mathbb{R}^d, \mathbf{G})$ on the domain $u \in H^1(\mathbb{R}^d, \mathbf{G})$. Let

$$V(\cdot) : \mathbb{R}^d \to B(\mathbf{G})$$

be an operator-valued function satisfying

$$\|V(\cdot)\|_{B(\mathbf{G})} \in L^p(\mathbb{R}^d) \tag{2.1}$$

for some finite p with

$$\begin{aligned} p &\geq 1 \quad \text{if} \quad d = 1, \\ p &> 1 \quad \text{if} \quad d = 2, \\ p &\geq d/2 \quad \text{if} \quad d \geq 3. \end{aligned}$$

Then the quadratic form

$$v[u,u] = \int_{\mathbb{R}^d} \langle Vu, u \rangle_{\mathbf{G}} \, dx$$

is bounded with respect to $h[\cdot,\cdot]$ and thus the form

$$h[u,u] + v[u,u] \tag{2.2}$$

is closed and semi-bounded from below on $H^1(\mathbb{R}^d, \mathbf{G})$. It generates the self-adjoint operator

$$Q = -(\Delta \otimes \mathbb{1}_{\mathbf{G}}) + V(x) \tag{2.3}$$

in $L^2(\mathbb{R}^d, \mathbf{G})$. It is not difficult to see, that if the operator $V(x)$ belongs to $\mathcal{K}(\mathbf{G})$ for a.e. $x \in \mathbb{R}^d$ and satisfies the condition (2.1), then the negative spectrum

$$-E_1 \leq -E_2 \leq \cdots \leq -E_n \leq \cdots < 0$$

of the operator Q is discrete.

3. An upper bound for the eigenvalue moment in the critical case $d = 1$ and $\gamma = 1/2$.

3.1. **A sharp Lieb-Thirring inequality for $d = 1$ and $\gamma = 1/2$.** In this section we give a version of the proof from [15] which will be applied to the Schrödinger operators with operator-valued potentials. The main result of this section is the following statement:

Theorem 3.1. *Let $V(x)$ be a nonpositive operator-valued function, such that $V(x) \in \mathfrak{S}_1(\mathbf{G})$ for a.e. $x \in \mathbb{R}$ and $\operatorname{tr} V_-(\cdot) \in L^1(\mathbb{R})$. Then*

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes 1_G + V\Big)_-^{1/2} = \sum_j \sqrt{E_j} \le \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{tr} V_-\, dx\,. \tag{3.1}$$

Remark. The constant $L_{1/2,1} = 1/2 = 2L^{\mathrm{cl}}_{1/2,1}$ is the best possible. Indeed, $1/2$ is achieved by the operator of rank one $V(x) = \delta(x)\langle\cdot, e\rangle\, e$, where $e \in \mathbf{G}$ and δ is Dirac's δ-function (see [15]).

We follow the strategy of [15] quite closely but give a different proof of the monotonicity lemma.

3.2. **Monotonicity Lemma.** In order to prove the monotonicity lemma we need an auxiliary "majorization" result. Let $A \in \mathcal{K}(\mathbf{G})$ and let us denote

$$\|A\|_n = \sum_{j=1}^{n} \sqrt{\lambda_j(A^*A)}.$$

Then by Ky-Fan's inequality (see for example [12, Lemma 4.2]) the functionals $\|\cdot\|_n$, $n = 1, 2, \dots$, are norms on $\mathcal{K}(\mathbf{G})$ and thus for any unitary operator U in $\mathbf{G}$ we have

$$\|U^*AU\|_n = \|A\|_n.$$

Definition 3.2. *Let A, B be two compact operators on $\mathbf{G}$. We say that A majorizes B or $B \prec A$, iff*

$$\|B\|_n \le \|A\|_n \qquad \textit{for all } n \in \mathbb{N}\,.$$

Lemma 3.3 (Majorization)**.** *Let A be a nonnegative compact operator $\mathbf{G}$, $\{U(\omega)\}_{\omega\in\Omega}$ be a family of unitary operators on $\mathbf{G}$, and let g be a probability measure on Ω. Then the operator*

$$B := \int_\Omega U^*(\omega) A U(\omega)\, g(d\omega)$$

is majorized by A.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n \le \int_\Omega \|U^*(\omega)AU(\omega)\|_n \, g(d\omega) = g(\Omega)\|A\|_n = \|A\|_n \,.$$

∎

Remark. The notion of majorization is well-known in matrix theory (see [3]). For finite dimensional Hilbert spaces $\mathbf{G}$ even the converse statement of Lemma 3.3 is true, cf. [2, Theorem 7.1]:
If A and B are nonnegative matrices and $\operatorname{tr} A = \operatorname{tr} B$, then the condition $B \prec A$ implies that there exist unitary matrices U_j and $t_j > 0, j = 1, \dots, N$, such that

$$\sum_{j=1}^{N} t_j = 1, \qquad B = \sum_{j=1}^{N} t_j U_j^* A U_j.$$

Let $W(\cdot) : \mathbb{R} \to \mathcal{S}_2(\mathbf{G})$ be an operator-valued function and let $\|W(\cdot)\|_{\mathcal{S}_2} \in L^2(\mathbb{R})$. Denote

$$\mathcal{L}_\varepsilon := W^* \left[2\varepsilon \Big(-\frac{d^2}{dx^2} + \varepsilon^2 \Big)^{-1} \otimes 1_{\mathbf{G}} \right] W \,. \tag{3.2}$$

Obviously, $\mathcal{L}_\varepsilon$ is a nonnegative, trace class operator on $L^2(\mathbb{R}, \mathbf{G})$, its trace is independent of ε, $0 \le \varepsilon < \infty$ and equals $\operatorname{tr} \mathcal{L}_\varepsilon = \int \|W(x)\|_{\mathcal{S}_2}^2 \, dx$.

Lemma 3.4 (Monotonicity)**.** *The operator $\mathcal{L}_\varepsilon$ is majorized by $\mathcal{L}_{\varepsilon'}$*

$$\mathcal{L}_\varepsilon \prec \mathcal{L}_{\varepsilon'}$$

for all $0 \le \varepsilon' \le \varepsilon$.

Proof. Using the majorization Lemma 3.3 the proof is basically reduced to a right choice of notation. Let A be the nonnegative compact operator in $L^2(\mathbb{R}, \mathbf{G})$, given by the integral kernel[1] $A(x, y) := W^*(x)W(y)$. Furthermore let

$$g_\varepsilon(dp) = \begin{cases} \varepsilon(\pi(p^2+\varepsilon^2))^{-1} \, dp & \text{if} \quad \varepsilon > 0 \\ \delta(dp) & \text{if} \quad \varepsilon = 0 \end{cases} \tag{3.3}$$

be the Cauchy distribution and $\{U(p)\}_{p \in \mathbb{R}}$ be the group of unitary multiplication operators $(U(p)\psi)(x) = e^{-ipx}\psi(x)$ on $L^2(\mathbb{R}, \mathbf{G})$. Passing to the Fourier representation of the Green function in (3.2) we obtain

$$\mathcal{L}_\varepsilon = \int_{-\infty}^{\infty} U^*(p) A U(p) \, g_\varepsilon(dp) \,. \tag{3.4}$$

[1]In the scalar case A would just be the rank one operator $|W\rangle\langle W|$ (in Dirac notation).

Of course, $\mathcal{L}_0 = A$. In particular, Lemma 3.3 and (3.4) immediately imply $\mathcal{L}_\varepsilon \prec \mathcal{L}_0$. The Cauchy distribution is a convolution semigroup, i.e. $g_\varepsilon = g_\varepsilon{}' * g_{\varepsilon-\varepsilon'}$. If we insert this into (3.4) and change variables using the group property of the unitary operators $U(p)$, then Lemma 3.3 yields

$$\mathcal{L}_\varepsilon = \int U^*(p)\mathcal{L}_{\varepsilon'}U(p)\, g_{\varepsilon-\varepsilon'}(p)dp \prec \mathcal{L}_{\varepsilon'}.$$

This completes the proof. ∎

3.3. **Proof of Theorem 3.1.** Let $W(x) = \sqrt{V_-(x)}$, so $W^* = W$. Then from the assumptions made in Theorem 3.1, we find that $W(x)$ is a family of nonnegative Hilbert-Schmidt operators such that $\|W(\cdot)\|_{\mathfrak{S}_2} \in L^2(\mathbb{R})$. Let

$$\mathcal{K}_E := \frac{1}{2\sqrt{E}}\mathcal{L}_{\sqrt{E}} = W\left[\left(-\frac{d^2}{dx^2}+E\right)^{-1}\otimes \mathbf{1}_G\right]W, \tag{3.5}$$

where $\mathcal{L}_\varepsilon$ is defined in (3.2). According to the Birman-Schwinger principle [4, 24] we have

$$1 = \lambda_j(\mathcal{K}_{E_j})$$

for all negative eigenvalues $\{-E_j\}_j$ of the Schrödinger operator (2.3). Multiplying this equality by $2\sqrt{E_j}$ and summing over j we obtain

$$2\sum \sqrt{E_j} = \sum \lambda_j(\mathcal{L}_{\sqrt{E_j}}). \tag{3.6}$$

In contrast to $\mathcal{K}_E$ the operator $\mathcal{L}_{\sqrt{E}}$ is well-behaved for small energies. We now use the same *monotonicity argument* as in [15] to dispose of the energy dependence of the operator in (3.6). Namely, for any $n \in \mathbb{N}$, Lemma 3.4 implies that the *partial traces* $\sum_{j\le n}\lambda_j(\mathcal{L}_\varepsilon)$ are *monotone decreasing* in ε. Given this monotonicity, a simple induction argument yields

$$\sum_{j\le n}\lambda_j(\mathcal{L}_{\sqrt{E_j}}) \le \sum_{j\le n}\lambda_j(\mathcal{L}_{\sqrt{E_n}}) \quad \text{for all } n\in\mathbb{N}.$$

Hence, by (3.6) we also have the bound

$$2\sum \sqrt{E_j} \le \sum \lambda_j(\mathcal{L}_0) = \operatorname{tr}\mathcal{L}_0 = \int_{-\infty}^{\infty} \operatorname{tr} W^2(x)\, dx = \int_{-\infty}^{\infty} \operatorname{tr} V_-(x)\, dx.$$

The proof is complete.

3.4. **A priori estimate for moments $\gamma \ge 1/2$.** Following Aizenman and Lieb [1] we can "lift" the bound of Theorem 3.1 to moments $\gamma \ge 1/2$.

Corollary 3.5. *Assume that* $V(x)$ *is a nonpositive operator-valued function for a.e.* $x \in \mathbb{R}$ *and that* $\operatorname{tr} V_-(\cdot) \in L^{\gamma+\frac{1}{2}}(\mathbb{R})$ *for some* $\gamma \geq 1/2$. *Then*

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{\gamma} = \sum_j E_j^{\gamma} \leq 2L^{\mathrm{cl}}_{\gamma,1}\int_{-\infty}^{\infty} \operatorname{tr} V_-^{\gamma+\frac{1}{2}}\, dx\,. \tag{3.7}$$

Proof. Note that Theorem 3.1 is equivalent to

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{1/2} \leq 2\iint_{\mathbb{R}\times\mathbb{R}} \operatorname{tr}(p^2 - V_-(x))_-^{1/2}\,\frac{dp\,dx}{2\pi}\,.$$

Scaling gives the simple identity for all $s \in \mathbb{R}$

$$s_-^{\gamma} = C_\gamma \int_0^\infty t^{\gamma-\frac{3}{2}}(s+t)_-^{1/2}\,dt\,, \qquad C_\gamma^{-1} = B\Big(\gamma - \frac{1}{2}, \frac{3}{2}\Big)\,,$$

where B is the Beta function. Let $\mu_j(x)$ the eigenvalues of $V_-(x)$. Then

$$\begin{aligned}
\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{\gamma} &= C_\gamma\int_0^\infty dt\, t^{\gamma-\frac{3}{2}} \operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V + t\Big)_-^{1/2} \\
&\leq C_\gamma \int_0^\infty dt\, t^{\gamma-\frac{3}{2}}\, 2\iint \operatorname{tr}(p^2 - V_- + t)_-^{1/2}\,\frac{dp\,dx}{2\pi} \\
&= 2\sum_{j=1}^{\infty}\iint \Big[C_\gamma\int_0^\infty dt\, t^{\gamma-\frac{3}{2}}(p^2-\mu_j+t)_-^{1/2}\Big]\,\frac{dp\,dx}{2\pi} \\
&= 2\iint \operatorname{tr}(p^2 - V_-)_-^{\gamma}\,\frac{dp\,dx}{2\pi} = 2\,L^{\mathrm{cl}}_{\gamma,1}\int \operatorname{tr} V_-^{\gamma+1/2}\,dx\,.
\end{aligned}$$

∎

4. New estimates on the constants $L_{\gamma,d}$ for $1/2 \leq \gamma < 3/2$, $d \in \mathbb{N}$

4.1. **The Main result.** We consider now the Schrödinger operator (2.3) in $L^2(\mathbb{R}^d, \mathbf{G})$ for an arbitrary $d \in \mathbb{N}$. Assume that V is a nonpositive operator-valued function satisfying the condition

$$\operatorname{tr} V(\cdot) \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d) \tag{4.1}$$

for some appropriate γ. We shall discuss bounds on the optimal constants in the Lieb-Thirring inequalities

$$\operatorname{tr}(-\Delta\otimes \mathbf{1} + V)_-^{\gamma} \leq L_{\gamma,d}\int_{\mathbb{R}^d} \operatorname{tr} V_-^{\frac{d}{2}+\gamma}\,dx\,. \tag{4.2}$$

In [17] it has been shown that

$$L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d} \qquad \text{for all} \quad \gamma \geq 3/2, \quad d \in \mathbb{N}\,. \tag{4.3}$$

The main result of the paper concerns $1/2 \leq \gamma < 3/2$.

Theorem 4.1. *Let* V *be a nonpositive operator-valued function and let the condition* (4.1) *be satisfied. Then the following estimates on the sharp constants* $L_{\gamma,d}$ *hold*

$$L_{\gamma,d} \le 2L^{cl}_{\gamma,d} \quad \textit{for all} \quad 1 \le \gamma < 3/2\,, \quad d \in \mathbb{N}\,, \tag{4.4}$$

$$L_{\gamma,d} \le 2L^{cl}_{\gamma,d} \quad \textit{for all} \quad 1/2 \le \gamma < 3/2\,, \quad d = 1\,, \tag{4.5}$$

$$L_{\gamma,d} \le 4L^{cl}_{\gamma,d} \quad \textit{for all} \quad 1/2 \le \gamma < 1\,, \quad d \ge 2\,. \tag{4.6}$$

Remark. For the special case $\gamma = 1$ we find that

$$L^{cl}_{1,d} \le L_{1,d} \le 2L^{cl}_{1,d} \quad \text{for all} \quad d \in \mathbb{N}\,.$$

Even in the scalar case $\mathbf{G} = \mathbb{C}$ this is a substantial improvement of the previously known numerical estimates on these constants in high dimensions obtained in [5] and [20].

Remark. In fact, our proof of Theorem 4.1 yields

$$L_{\gamma,d} \le \frac{L_{\gamma,1}}{L^{cl}_{\gamma,1}} L^{cl}_{\gamma,d}, \qquad d \in \mathbb{N}, \quad 1 \le \gamma < 3/2\,.$$

According to Corollary 3.5 we know that $L_{1,1} \le 2L^{cl}_{1,1}$. In the scalar case Lieb and Thirring conjectured that

$$\frac{L_{\gamma,1}}{L^{cl}_{\gamma,1}} = 2\left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma-1/2}, \qquad 1/2 \le \gamma < 3/2\,.$$

In particular, if this were true in the matrix case for $\gamma = 1$, our approach would imply $L^{cl}_{1,1} \le L_{1,d} < 1.16\, L^{cl}_{1,d}$.

Proof of Theorem 4.1. We apply an induction argument similar to the one used in [17]. For $d = 1$ and $1/2 \le \gamma < 3/2$ the bound (4.5) is identical to (3.7).

Consider the operator (2.3) in the (external) dimension d. We rewrite the quadratic form $h[u,u] + v[u,u]$ for $u \in H^1(\mathbb{R}^d, \mathbf{G})$ as

$$h[u,u] + v[u,u] = \int_{-\infty}^{+\infty} h(x_d)[u,u]\, dx_d + \int_{-\infty}^{+\infty} w(x_d)[u,u]\, dx_d\,,$$

$$h(x_d)[u,u] = \int_{\mathbb{R}^{d-1}} \left\| \frac{\partial u}{\partial x_d} \right\|^2_{\mathbf{G}} dx_1 \cdots x_{d-1}\,,$$

$$w(x_d)[u,u] = \int_{\mathbb{R}^{d-1}} \left[\sum_{j=1}^{d-1} \left\| \frac{\partial u}{\partial x_j} \right\|^2_{\mathbf{G}} + \langle V(x)u, u\rangle_{\mathbf{G}} \right] dx_1 \cdots x_{d-1}\,.$$

The form $w(x_d)$ is closed on $H^1(\mathbb{R}^{d-1}, \mathbf{G})$ for a.e. $x_d \in \mathbb{R}$ and it induces the self-adjoint operator

$$W(x_d) = -\sum_{k=1}^{d-1} \frac{\partial^2}{\partial x_k^2} \otimes \mathbf{1}_{\mathbf{G}} + V(x_1, \dots, x_{d-1}; x_d)$$

on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$. For a fixed $x_d \in \mathbb{R}$ this is a Schrödinger operator in $d-1$ dimensions. Its negative spectrum is discrete, hence $W_-(x_d)$ is compact on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$.

Assume that we have (4.4)–(4.5) for the dimension $d-1$ and all γ from the interval $1/2 \leq \gamma < 3/2$. Then $\operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x_d)$ satisfies the bound

(4.7)

$$\operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x_d) \leq L_{\gamma+\frac{1}{2},d-1} \int_{\mathbb{R}^{d-1}} \operatorname{tr} V_-^{\gamma+\frac{d}{2}}(x_1, \dots, x_{d-1}; x_d)\, dx_1 \cdots dx_{d-1}$$

for a.e. $x_d \in \mathbb{R}$. Here

$$L_{\gamma+\frac{1}{2},d-1} = L^{\mathrm{cl}}_{\gamma+\frac{1}{2},d-1} \qquad \text{for} \qquad \gamma \geq 1\,, \tag{4.8}$$

$$L_{\gamma+\frac{1}{2},d-1} \leq 2L^{\mathrm{cl}}_{\gamma+\frac{1}{2},d-1} \qquad \text{for} \qquad 1/2 \leq \gamma < 1\,. \tag{4.9}$$

Indeed, (4.8) follows from (4.3) and (4.9) follows from (4.4)–(4.5) in dimension $d-1$.

Let $w_-(x_d)[\cdot,\cdot]$ be the quadratic form corresponding to the operator $W_-(x_d)$ on $\mathbf{H} = L^2(\mathbb{R}^{d-1}, \mathbf{G})$. We have $w(x_d)[u,u] \geq -w_-(x_d)[u,u]$ and

$$h[u,u] + v[u,u] \geq \int_{-\infty}^{+\infty} \left[\left\| \frac{\partial u}{\partial x_d} \right\|_{\mathbf{H}}^2 - \langle W_-(x_d)u, u\rangle_{\mathbf{H}} \right] dx_d \tag{4.10}$$

for all $u \in H^1(\mathbb{R}^d, \mathbf{G})$. According to section 2.2 the form on the r.h.s. of (4.10) can be closed to $H^1(\mathbb{R}, \mathbf{H})$ and induces the self-adjoint operator

$$-\frac{d^2}{dx_d^2} \otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d)$$

on $L^2(\mathbb{R}, \mathbf{H})$. Then (4.10) implies

$$\operatorname{tr}(-\Delta \otimes \mathbf{1}_{\mathbf{G}} + V)_-^{\gamma} \leq \operatorname{tr} \left(-\frac{d^2}{dx_d^2} \otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d) \right)_-^{\gamma}. \tag{4.11}$$

The assumption $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ implies that $\operatorname{tr} W_-^{\gamma+\frac{1}{2}}$ is an integrable function and we can apply Corollary 3.5 to the r.h.s. of (4.11). In view of (4.7) we find

$$\operatorname{tr}\left(-\frac{d^2}{dx_d^2}\otimes 1_H - W_-(x_d)\right)_-^{\gamma} \le L_{\gamma,1}\int_{-\infty}^{+\infty} \operatorname{tr} W_-^{\gamma+\frac12}(x_d)\,dx_d$$
$$\le L_{\gamma,1}L_{\gamma+\frac12,d-1}\int_{\mathbb{R}^d}\operatorname{tr} V_-^{\gamma+\frac d2}\,dx$$

for $\gamma \ge 1/2$. The bounds (4.5), (4.8) or (4.9) and the calculation

$$L^{cl}_{\gamma,1}L^{cl}_{\gamma+\frac12,d-1} = \frac{\Gamma(\gamma+1)}{2\pi^{\frac12}\Gamma(\gamma+\frac12+1)}\cdot\frac{\Gamma(\gamma+\frac12+1)}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma(\gamma+\frac12+\frac{d-1}{2}+1)}$$
$$= \frac{\Gamma(\gamma+1)}{2^d\pi^{\frac d2}\Gamma(\gamma+\frac d2+1)} = L^{cl}_{\gamma,d}$$

complete the proof. ∎

4.2. **Estimates for magnetic Schrödinger operators.** Following a remark by B. Helffer [13] and using the arguments from [17] we can extend Theorem 4.1 to Schrödinger operators with magnetic fields. Let $Q(a)$ be a self-adjoint operator in $L^2(\mathbb{R}^d, \mathbf{G})$

$$Q(a) = (i\nabla + a(x))^2 \otimes 1_G + V(x), \tag{4.12}$$

where

$$a(x) = (a_1(x), \cdots, a_d(x))^t\,, \qquad d \ge 2\,,$$

is a magnetic vector potential with real-valued entries $a_k \in L^2_{loc}(\mathbb{R}^d)$.

We consider the inequality

$$\operatorname{tr}(Q(a))_-^{\gamma} \le \tilde{L}_{\gamma,d}\int_{\mathbb{R}^d} V_-^{\frac d2+\gamma}\,dx\,, \tag{4.13}$$

where the nonpositive operator function $V(\cdot)$ satisfies (4.1). In [17] it has been shown, that

$$\tilde{L}_{\gamma,d} = L^{cl}_{\gamma,d} \qquad \text{for all} \qquad \gamma \ge 3/2\,, \quad d \in \mathbb{N}. \tag{4.14}$$

In general, the sharp constant $\tilde{L}_{\gamma,d}$ in (4.14) might differ from the sharp constant $L_{\gamma,d}$ in (4.2)

$$L^{cl}_{\gamma,d} \le L_{\gamma,d} \le \tilde{L}_{\gamma,d}\,.$$

By combining the arguments from [17] and those used in the prove of Theorem 4.1 we immediately obtain the following result:

Theorem 4.2. *The following estimates on the sharp constants* $\tilde{L}_{\gamma,d}$ *in* (4.13) *hold*

(4.15) $$\tilde{L}_{\gamma,d} \leq 2L^{cl}_{\gamma,d} \quad \textit{for all} \quad 1 \leq \gamma < 3/2\,, \quad d \geq 2\,,$$

(4.16) $$\tilde{L}_{\gamma,d} \leq 4L^{cl}_{\gamma,d} \quad \textit{for all} \quad 1/2 \leq \gamma < 1\,, \quad d \geq 2\,.$$

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