ON THE NEGATIVE EIGENVALUES OF A CLASS OF SCHRÖDINGER OPERATORS

BY

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This paper is dedicated to M.S. Birman on the occasion of his 70th birthday with our warmest wishes and gratitude

ABSTRACT. We find a subclass of potentials satisfying the CLR type inequalities for the number of negative eigenvalues of the operator $(-\Delta)^l + |x|^{-2l} - V$, $l \in \mathbb{N}$, in $\mathbb{R}^d$ for the limiting case when $d = 2l$.

1. INTRODUCTION

1.1. CLR-type inequalities. We study the selfadjoint operator in $L^2(\mathbb{R}^d)$

$$H = H_V = (-\Delta)^l + b|x|^{-2l} - V, \qquad l \in \mathbb{N}, \quad b \in \mathbb{R}, \tag{1.1}$$

where V is a nonnegative, locally integrable function (potential) in $\mathbb{R}^d$. The operator (1.1) can be accurately defined by its quadratic form. Denote by $N_b(V)$ the number of negative eigenvalues of the operator (1.1).

If $2l < d$ and $V \in L^{d/2l}(\mathbb{R}^d)$, then for any $b > -((d-2)\ldots(d-2l))^2/2^{2l}$ the following inequality holds

$$N_b(V) \leqslant C(b,d,l) \int V^{d/2l}\, dx. \tag{1.2}$$

For $l = 1$ (1.2) is known as the Cwikel-Lieb-Rozenblum (CLR) inequality.

If $2l \geqslant d$ and $b \geqslant 0$, then the inclusion $V \in L^{d/2l}(\mathbb{R}^d)$ does not imply (1.2). In fact, this inclusion does not guarantee even the semiboundedness of the operator (1.1) from below. For $2l > d$ some different type estimates of the number of the negative eigenvalues were obtained in [BS1] for d odd and in [BLS] for d even. In the case $2l = d$ the corresponding results are less complete (see [BL], [BLS], [L] and [S1,2]). It was first shown in [S1,2], and then in a sharper form in [BL] and [BLS], that if $2l = d$ and $b = 0$, then the problem can be separated into two problems. The first one is defined by the restriction of the operator (1.1) to the subspace of functions depending on $|x|$ and, hence, is reduced to a well studied one-dimensional

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differential operator with the potential equal to the mean value $\widetilde{V}$ of V over $\mathbb{S}^{d-1}$. In particular, for this class of operators there are (see [BS1]) necessary and sufficient conditions on the potential $\widetilde{V}$ which give $N_0(\alpha\widetilde{V}) = O(\alpha)$, as $\alpha \to \infty$. The second problem is defined by a class of functions whose mean values over $\mathbb{S}^{d-1}$ are equal to zero. On this subspace we have the Hardy inequality which automatically provides us with the "supporting" term $b|x|^{-2l}$ with some $b > 0$. This suggests that in order to study the case $2l = d$, we have to pay special attention to the operator (1.1), where $b > 0$.

The purpose of this paper is to find a subclass of potentials from $L^1(\mathbb{R}^d)$, such that the inequality (1.2) holds for $d = 2l$ and $b > 0$. We shall always assume that $b = 1$. For an arbitrary $b > 0$ all the statements of this paper remain true but the constants depend on b. For $b = 1$ (1.2) takes the form

$$N(V) = N_1(V) \leqslant C(d) \int V(x)\, dx. \tag{1.3}$$

The right hand side of (1.3) does not require more than $V \in L^1(\mathbb{R}^d)$. We prefer to deal with the problem

$$H_\mu = (-\Delta)^l + |x|^{-2l} - \mu, \tag{1.4}$$

where μ is a nonnegative, finite measure in $\mathbb{R}^d$. If μ is an absolutely continuous measure, $d\mu = V dx$, and $b = 1$, then (1.4) coincides with (1.1). Let $|\nabla^l u|^2 := \sum_{|\beta|=l}(l!/\beta!)|\partial^\beta u|^2$. We shall impose such conditions on μ that the quadratic form

$$h_\mu[u,u] = \int (|\nabla^l u|^2 + |x|^{-2l}|u|^2)\, dx - \int |u|^2\, d\mu \tag{1.5}$$

defined on $\mathcal{H}^l(\mathbb{R}^d)$ (see(1.8)) is semibounded and closed in $L^2(\mathbb{R}^d)$ and, hence, defines a selfadjoint operator H_μ. Notice that necessary and sufficient conditions of closability and semiboundedness of a wide classes of quadratic forms were obtained in [M, Ch.8 and 12].

For $b = 0$ the operator (1.1) has already been studied in [S1,2], where some estimates of $N_0(V)$ were obtained in terms of Orlicz classes. This paper deals with the problem of finding a class of potentials, such that the prescribed inequality (1.3) is satisfied. Our conditions are different, and the results of this paper and those obtained in [S1,2] complement each other. In particular, if $d\mu = V dx$ and $V(x) = V(|x|) \in L^1(\mathbb{R}^d)$, $d = 2l$, then our results imply the inequality (1.3) (see also [L]).

1.2. The main results. In order to formulate the main result we introduce the following definition and notation.

The open ball with centre at $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by $B(x,r)$,

$$B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}.$$

Condition $(*)$**.** Let μ be a nonnegative measure in $\mathbb{R}^d$, $d \geqslant 1$, whose support $F = \operatorname{supp}\mu$ is a bounded set. We say that the measure μ satisfies Condition $(*)$ with constants γ_1 and γ_2, where $\gamma_1, \gamma_2 > 1$, if for any $x \in \mathbb{R}^d$ and $r \leqslant \gamma_2^{-1}\operatorname{diam} F$ we have that

$$\mu(B(x,\gamma_1 r)) \geqslant 2\mu(B(x,r))$$

Denote

$$\Omega_{c_1,c_2} := \{x \in \mathbb{R}^d : c_1 \leqslant |x| < c_2\} = B(0,c_2) \setminus B(0,c_1), \qquad 0 \leqslant c_1 < c_2 \leqslant \infty,$$

$$\Omega_k := \Omega_{c_1,c_2}, \quad \text{where} \quad c_1 = 2^{k-1}, \quad c_2 = 2^k, \quad k \in \mathbb{Z}$$

and

$$\widetilde{\Omega}_k := \Omega_{c_1,c_2}, \quad \text{where} \quad c_1 = 2^{k-2}, \quad c_2 = 2^{k+1}, \quad k \in \mathbb{Z}.$$

Our main result is the following statement:

Theorem 1.1. *Let μ be a finite, nonnegative Borel measure in $\mathbb{R}^d$, d even, whose restrictions $\mu|_{\overline{\Omega}_k}$, $k \in \mathbb{Z}$ satisfy Condition $(*)$ with constants γ_1 and γ_2 independent of k. Then the quadratic form (1.5) is semibounded from below, closed on $\mathcal{H}^l(\mathbb{R}^d)$ and the number of negative eigenvalues $N(\mu)$ of the corresponding operator*

$$H_\mu = (-\Delta)^l + |x|^{-2l} - \mu, \qquad l = d/2 \in \mathbb{N}, \tag{1.6}$$

satisfies

$$N(\mu) \leqslant C\,\mu(\mathbb{R}^d), \tag{1.7}$$

where $C = C(\gamma_1, \gamma_2, d)$.

The proof of this theorem of is given in Section 4.

Remark 1.1. It can be easily checked (see Section 5) that Condition $(*)$ is satisfied for any spherically symmetric measure. In particular, if μ is the δ-function of $\mathbb{S}^{d-1}$, then the constants γ_1 and γ_2 can for example, be chosen as $\gamma_1 = \gamma_2 = 2$. In fact, Condition $(*)$ is satisfied for the δ-function of an arbitrary compact smooth submanifold of $\mathbb{R}^d$ of a positive dimension.

The next result is related to absolutely continuous measures $\mu = V\,dx$. Its proof follows from Theorem 1.1, but requires some additional technical preparations (see Section 5). Let us introduce a class of functions $L^1(\mathbb{R}_+, L^p(\mathbb{S}^{d-1}))$ defined in polar coordinates in $\mathbb{R}^d$, $x = (r,\theta)$, $r \in \mathbb{R}_+ = (0,\infty)$, as

$$\|f\|_{L^1(\mathbb{R}_+,\, L^p(\mathbb{S}^{d-1}))} = \int_0^\infty \Big(\int_{\mathbb{S}^{d-1}} |f(r,\theta)|^p\,d\theta\Big)^{1/p} r^{d-1}\,dr < \infty.$$

Theorem 1.2. *Let d be even, $l = d/2$, $V \geqslant 0$ and $V \in L^1(\mathbb{R}_+, L^p(\mathbb{S}^{d-1}))$, $1 < p \leqslant \infty$. Then the number of the negative eigenvalues of the operator (1.1) with $b = 1$ satisfies*

$$N(V) \leqslant C\|V\|_{L^1(\mathbb{R}_+,\, L^p(\mathbb{S}^{d-1}))},$$

where $C = C(d,p)$.

In particular, we immediately obtain the following

Corollary 1.3. *Let $l = d/2 \in \mathbb{N}$, $V \geqslant 0$ and $V(x) = V(|x|)$. If $V \in L^1(\mathbb{R}^d)$, then the inequality (1.3) is fulfilled.*

Remark 1.2. Even for the class of spherically symmetric potentials the inequality (1.3) fails if we do not introduce the "supporting" term $|x|^{-2l}$. Indeed, as it was shown in [BL] and in [BLS], if $\mu = V(x)dx$ and V is a smooth potential, such that

$$V(x) \sim |x|^{-2l} \ln^{-2}|x| (\ln\ln|x|)^{-1/q}, \qquad \text{as} \quad |x| \to \infty, \quad q > 1,$$

then the number of negative eigenvalues of the operator $(-\Delta)^l - V$ satisfies the following asymptotic formula

$$N_0(\alpha V) = \alpha^q c_q + o(\alpha^q), \qquad \alpha \to \infty,$$

although $V \in L^1(\mathbb{R}^d)$ for any $q > 0$. This is in contrast with (1.3).

1.3. Some notation. We shall denote by $L^p(\mathbb{R}^d,\mu)$ the class of L^p-integrable functions with respect to a measure μ. If μ coincides with the Lebesgue measure, then we omit μ and write $L^p(\mathbb{R}^d,\mu)=L^p(\mathbb{R}^d)$. Let G be an open subset of R^d. By $H^l(G)$ we denote the Sobolev class of the order l equipped by the standard Hilbert metric

$$H^l(G)=\Big\{f:\int_G\Big(|\nabla^l f|^2+|f|^2\Big)\,dx<\infty\Big\}.$$

The integral over the whole space is written without indicating the domain of integration. The class of functions $\mathcal{H}^l(\mathbb{R}^d)$, $l=d/2$, $l\in\mathbb{N}$, is a so-called homogeneous H^l class and defined by

$$\mathcal{H}^l(\mathbb{R}^d)=\Big\{f:\int\Big(|\nabla^l f|^2+\frac{|f|^2}{|x|^{2l}}\Big)\,dx<\infty\Big\}. \tag{1.8}$$

C and c will be different constants whose values are unimportant. By $\mathcal{P}^l$ is denoted the class of polynomials in $\mathbb{R}^d$ of degree less than or equal to l. By v_d we denote the volume of the unit ball in $\mathbb{R}^d$,

$$v_d:=|B(0,1)|=\mathrm{vol}\{x\in\mathbb{R}^d:\,|x|<1\}=\frac{\pi^{d/2}}{\Gamma(1+d/2)}.$$

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2. Covering Lemmas

Let us first recall a classical result of Besicovitch [B1,2] (see also [G], Ch.1)

Lemma 2.1. *Let $A\subset\mathbb{R}^d$, $d\geqslant 1$, be a compact set and r be a positive function on A. Then there exists a finite subset $J\subset A$ and a family of balls $\{B(x,r(x))\}_{x\in J}$, such that the following two conditions are fulfilled:*

(1) $\cup_{x\in J}B(x,r(x))\supset A$,

(2) *for any $y\in A$*

$$\#\{x:\,x\in J\ \text{ and }\ \overline{B}(x,r(x))\ni y\}\leqslant C,$$

where the constant $C=C(d)$ depends only on the dimension d.

Lemma 2.1 was first applied for the problem of spectral estimates in [BS2]. The next result follows from Lemma 2.1 and already appeared in [R] for absolutely continuous measures. It will be used in the proof of Theorem 1.1.

Lemma 2.2. *Let μ be a finite, continuous, nonnegative Borel measure in $\mathbb{R}^d$. Suppose that its support $F=\operatorname{supp}\mu$ is a bounded set. Then for any $m\in\mathbb{N}$ there exists a finite set $J\subset F$ and a family of balls $\{B(x,r(x))\}_{x\in J}$ satisfying the following conditions:*

(1) $\cup_{x\in J}B(x,r(x))\supset F$,

(2) $\mu(B(x,r(x)))\leqslant\frac{\mu(\mathbb{R}^d)}{m}\leqslant\mu(\overline{B}(x,r(x))),\quad x\in J$,

(3) *for any $y\in\mathbb{R}^d$*

$$\#\{x:\,x\in J\ \text{ and }\ y\in\overline{B}(x,r(x))\}\leqslant C_1(d),$$

(4) $\#\{x:\,x\in J\}\leqslant C_2(d)\,m$. *Here the constants C_1 and C_2 depend only on the dimension d.*

Proof. For an arbitrary $x \in F$ and $m \in \mathbb{N}$ define

$$\widetilde{r}(x) = \sup\Big\{r(x) \in \mathbb{R}_+ : \mu(B(x, r(x)) \leqslant \frac{\mu(\mathbb{R}^d)}{m}\Big\}. \tag{2.1}$$

Then

$$\mu(B(x, \widetilde{r}(x))) \leqslant \frac{\mu(\mathbb{R}^d)}{m}, \qquad x \in F,$$

and obviously

$$\mu(\overline{B}(x, \widetilde{r}(x))) \geqslant \frac{\mu(\mathbb{R}^d)}{m}, \qquad x \in F. \tag{2.2}$$

Applying now Lemma 2.1 we find a finite set J such that (1)–(3) are fulfilled. Since the support of F is a compact set, then (3) and (2.2) imply (4). The proof is complete. □

Remark 2.1. If $\operatorname{supp}\mu \subset \overline{\Omega}_1 = \overline{B(0,2)} \setminus B(0,1)$, then in Lemma 2.2 we could choose the family of covering balls $\{B(x, r(x))\}$, $x \in J$, such that their supports were lying in $\operatorname{Int}\widetilde{\Omega}_1 = \operatorname{Int}(B(0,4) \setminus B(0,1/2))$. Indeed, when introducing $\widetilde{r}(x)$, we could require in addition that the supremum is taken over $r(x) \leqslant 1/2$. Then the proof of the conditions (1)-(3) remains the same. The estimate for the number of points $x \in J$ satisfying (4) with $\widetilde{r}(x) < 1/2$ is the same, but the number of balls with $\widetilde{r}(x) = 1/2$ is bounded.

3. Some inequalities from real analysis

We collect here preliminary material which prevents us from being distracted while proving the main result.

The next statement is a version of the well-known Poincaré inequality (see, for example, [M, Ch.1.1.11] or [AH, Ch.8.1]).

Lemma 3.1. *For any $l \in \mathbb{N}$ and any ball $B(0,r) \subset \mathbb{R}^d$, $r > 0$, there exists a linear operator (orthogonal projection in $L^2(B(0,r))$*

$$T : \ L^2(B(0,r)) \hookrightarrow \mathcal{P}^{l-1}, \tag{3.1}$$

such that

$$\|f - Tf\|^2_{L^2(B(0,r))} \leqslant C(d)\, r^{2l}\, \|\nabla^l f\|^2_{L^2(B(0,r))}.$$

The proof of the following statement is due to Adams [A] (see also [M, Ch.8] and [AH, Th. 7.2.2] where there are also many other related results). It concerns Sobolev spaces H^α of any (not necessarily integer) positive order α.

Theorem 3.2. *Let $\alpha < d/2$ and μ be a finite, nonnegative measure in $\mathbb{R}^d$, $F = \operatorname{supp}\mu \subset B(0,1)$ and suppose that there is a constant C_*, such that for some $\beta > 0$ and any r, $0 < r < \infty$*

$$\mu(B(x,r)) \leqslant C_* r^\beta, \qquad x \in F.$$

If $\frac{\beta}{p} = \frac{d}{2} - \alpha$, $p > 2$, then the embedding operator

$$H^\alpha(B(0,1)) \hookrightarrow L^p(B(0,1), \mu)$$

is bounded and its norm does not exceed $c\, C_*^{1/p}$, where $c = c(\alpha, d, \beta)$ is independent of the measure μ.

In particular, this theorem implies the following weaker result for the case $2\alpha =$

Corollary 3.3. *Let $l = d/2$ and μ be a finite, non-negative measure in $\mathbb{R}^d$, $F = \operatorname{supp}\mu \subset B(0,1)$ and suppose there exists a constant C_*, such that for some $\beta > 0$ and any r, $0 < r < \infty$,*

$$\mu(B(x,r)) \leqslant C_* r^\beta, \qquad x \in F.$$

Then the embedding operator

$$H^l(B(0,1)) \hookrightarrow L^2(B(0,1),\mu)$$

is bounded and its norm does not exceed $C\,C_*^{1/2}$, where $C = C(l,\beta)$.

Proof. The proof is very simple. Choose $p > 1$ and α', such that $l > \alpha' = d/2 - \beta/2p > 0$. Applying the Hölder inequality with $q = p/(p-1)$ and Theorem 3.2 we find

$$\begin{aligned}\|u\|_{L^2(B(0,1),\mu)} \leqslant \|u\|_{L^{2p}(B(0,1),\mu)}\ \mu^{1/2q}(B(0,1)) \leqslant cC_*^{1/2p}\|u\|_{H^{\alpha'}(B(0,1))}\ C_*^{1/2q} \\ \leqslant CC_*^{1/2}\|u\|_{H^l(B(0,1))}.\end{aligned}$$

This completes the proof. $\square$

Using dilation and Corollary 3.3 we obtain (see [M, Lemma 1.4.7])

Lemma 3.4. *Let $l = d/2 \in \mathbb{N}$, $\beta > 0$, μ nonnegative, finite measure in $\mathbb{R}^d$ and $\operatorname{supp}\mu \subset \overline{B}(0,r)$, $r > 0$. Then there exists a constant $C = C(\beta,d)$, such that*

$$\|f\|^2_{L^2(B(0,r),\mu)} \leqslant C\, r^\beta\, M\left(\|\nabla^l f\|^2_{L^2(B(0,r))} + r^{-d}\|f\|^2_{L^2(B(0,r))}\right), \tag{3.2}$$

where

$$M = \sup_{x\in\mathbb{R}^d,\,\rho>0} \frac{\mu(B(x,\rho))}{\rho^\beta} \tag{3.3}$$

and $C = C(l,\beta)$.

Lemmas 3.1 and 3.4 immediately give us

Corollary 3.5. *Let T be the orthogonal projection defined in (3.1), $l = d/2 \in \mathbb{N}$, $\beta > 0$, let μ be a nonnegative, finite measure in $\mathbb{R}^d$ and $\operatorname{supp}\mu \subset \overline{B}(0,r)$, $r > 0$. Then there exists a constant $C = C(\beta,d)$, such that*

$$\|f - Tf\|^2_{L^2(B(0,r),\mu)} \leqslant C\, r^\beta\, M\, \|\nabla^l f\|^2_{L^2(B(0,r))}, \tag{3.4}$$

where M is given by (3.3) and $C = C(l,\beta)$.

Remark 3.1. When proving the next lemma we use the following simple remark: if μ satisfies Condition $(*)$ with the constants (γ_1,γ_2), then there are constants $\alpha > 0$ and $\varkappa > 0$ such that for any $0 < r < \infty$, $\gamma > 1$ and $\gamma r \leqslant \gamma_1 \operatorname{diam} F/\gamma_2$ we have

$$\mu(B(x,\gamma r)) \geqslant \varkappa\gamma^\alpha\mu(B(x,r)), \qquad x \in \mathbb{R}^d. \tag{3.5}$$

For example, we can take $\alpha = \ln 2/\ln \gamma_2$ and $\varkappa = 1/2$.

Let $m, L \in \mathbb{N}$, $A > 1$ be constants satisfying the inequalities $A^L/m \leqslant 1 < A^{L+1}/m$ and let $C_1(d)$ be the constant appearing in Lemma 2.1. By using Lemma 2.2 we choose a family of balls $B_{v,j} = B(x_{vj}, r_{vj})$, $x_{vj} \in \mathbb{R}^d$, $r_{vj} > 0$, $v = 0, 1, \dots, L$, $j = 1, 2, \dots, n_v$, so that for any $v = 0, 1, \dots, L$ the following conditions are fulfilled:

$$n_v \leqslant C(A, d)\, m\, A^{-v}, \tag{3.6}$$

$$F \subset \cup_{j=1}^{n_v} B_{v,j}, \quad \#\{j : 1 \leqslant j \leqslant n_v,\, y \in \overline{B}_{v,j}\} \leqslant C_1(d)$$

for any $y \in \mathbb{R}^d$ and

$$\mu(B_{v,j}) \leqslant \frac{A^v \mu(\mathbb{R}^d)}{m} \leqslant \mu(\bar{B}_{v,j}), \quad j = 1, 2, \dots, n_v.$$

Let $\Lambda = \Lambda(d)$ denote the maximum number of balls with the following properties: i) radii of the balls do not exceed $1/2$; ii) all the balls intersect $B(0, 1)$; iii) any point $x \in \mathbb{R}^d$ belongs to not more than $C_1(d)$ balls.

Lemma 3.6. *Let μ be a finite, nonnegative measure in R^d, $F = \operatorname{supp} \mu$ be a bounded set and μ satisfies Condition $(*)$ with the constants (γ_1, γ_2). Let A, m, L, $\Lambda(d)$ be the constants and $\{B_{v,j}\}_{j=1}^{n_v}$, $v = 0, 1, \dots, L$, be the families of balls introduced above. Then for any ball $B(x, r)$ satisfying*

$$\mu(B(x, r)) \geqslant \frac{K\mu(\mathbb{R}^d)}{m},$$

there exists a ball $B_{u,i} = B(x_{ui}, r_{ui})$, $0 \leqslant u \leqslant L-1$, $1 \leqslant i \leqslant n_u$, with the properties

$$|x - x_{ui}| \leqslant 3r \quad \text{and} \quad \frac{r}{2^u} \geqslant r_{ui} \geqslant \frac{r}{\zeta 2^u},$$

where ζ is defined by $(\zeta/2)^\alpha \varkappa = KA$ and $K > \max(\Lambda(d), A)$.

Proof. From the assumptions

$$K > \Lambda(d), \quad \mu(B(x, r)) \geqslant \frac{K\mu(\mathbb{R}^d)}{m}, \quad \mu(B_{0,j}) \leqslant \frac{\mu(\mathbb{R}^d)}{m}, \quad j = 1, 2, \dots, n_0,$$

it follows that there exists j_0 such that $B_{0,j_0} \cap B(x, r) \neq \varnothing$ and $r_{0j_0} < r/2$. If $r_{0j_0} \geqslant r/\zeta$, then the statement of the lemma is fulfilled if we take $B_{u,i} = B_{0,j_0}$. Thus we can assume that $r_{0j_0} < r/\zeta$. Let us introduce a new ball $B_1 = B(x_1, r/2) = B(x_{0j_0}, r/2)$. Then (3.5) implies

$$\mu(B_1) \geqslant \varkappa\Big(\frac{r/2}{r/\zeta}\Big)^\alpha \mu(\overline{B}_{0,j_0}) > \varkappa\Big(\frac{\zeta}{2}\Big)^\alpha \frac{\mu(\mathbb{R}^d)}{m} = \frac{K\,A\,\mu(\mathbb{R}^d)}{m}$$

and

$$|x_1 - x| \leqslant r + r/2.$$

At the next step we repeat our arguments for the ball $B(x_1, r/2)$ instead of $B(x, r)$ and the family $\{B_{1,j}\}_{j=1}^{n_1}$ instead of $\{B_{0,j}\}_{j=1}^{n_0}$. By using the inequalities

$$K > \Lambda(d), \quad \mu(B(x_1, r/2)) \geqslant \frac{AK\mu(\mathbb{R}^d)}{m},$$

$$\mu(B_{1,j}) \leqslant \frac{A\mu(\mathbb{R}^d)}{m}, \quad j = 1, 2, \ldots, n_1,$$

we find j_1 such that $B(x_1, r/2) \cap B(x_{1j_1}, r_{1j_1}) \neq \varnothing$ and $r_{1j_1} < r/4$. If $r_{1j_1} > r/2\zeta$, then the statement of the lemma is fulfilled if we take $B_{u,i} = B_{1,j_1}$. Thus we can assume that $r_{1j_1} < r/2\zeta$ and introduce $B_2 = B(x_2, r/4) = B(x_{1j_1}, r/4)$. Then by again applying (3.5) we obtain

$$\mu(B_2) \geqslant \varkappa \left(\frac{\zeta}{2}\right)^\alpha \frac{A\,\mu(\mathbb{R}^d)}{m} \geqslant \frac{K\,A^2\,\mu(\mathbb{R}^d)}{m}$$

and

$$|x_1 - x| \leqslant |x_1 - x| + |x_2 - x_1| \leqslant r + r/2 + r/2 + r/4.$$

Continuing this process we either find a ball $B_{u,i}$, $0 \leqslant u \leqslant L-1$, $1 \leqslant i \leqslant n_u$, satisfying the statement of the lemma or arrive at a ball B_L with the property

$$\mu(B_L) \geqslant \frac{KA^L}{m}\mu(\mathbb{R}^d) > \frac{A^{L+1}}{m}\mu(\mathbb{R}^d).$$

The last inequality is impossible since $A^{L+1}/m > 1$ and, therefore, the proof is complete. $\square$

Let $x_0 \in \mathbb{R}^d$, $0 < r_0 < \infty$ and $0 < \beta < \alpha$. Denote

$$\varphi(B(x_0, r_0)) = r_0^\beta \sup_{x \in B(x_0, r_0),\, 0 < r < r_0} r^{-\beta} \mu\Big(B(x, r) \cap B(x_0, r_0)\Big). \tag{3.7}$$

Correspondingly the value $\varphi(\overline{B}(x_0, r_0))$ is defined by (3.7), where the open ball $B(x, r)$ is changed by $\overline{B}(x, r)$.

Lemma 3.7. *Let $l = d/2 \in \mathbb{N}$, μ be a nonnegative, finite measure satisfying Condition $(*)$ with the constants (γ_1, γ_2) and $\operatorname{supp}\mu \subset \overline{\Omega}_1$. Then for any $m \in \mathbb{N}$ there exists a subspace $E \subset H^l(\widetilde{\Omega}_1)$, such that $\dim E \leqslant C \cdot m$ and for any $f \in H^l(\widetilde{\Omega}_1)$, $f \perp E$ we have*

$$\int_{\overline{\Omega}_1} |f(x)|^2\, d\mu(x) \leqslant C' \, \frac{\mu(\overline{\Omega}_1)}{m} \int_{\widetilde{\Omega}_1} |\nabla^l f(x)|^2\, dx,$$

where $C' = C'(d, \gamma_1, \gamma_2)$, $C = C(d, \gamma_1, \gamma_2)$.

Proof. Let us assume that we can find a family of balls $\{B_k\}_{k=1}^S$ satisfying the properties:

$$S \leqslant C_0\, m, \tag{3.8}$$

$$F \subset \cup_{k=1}^S B_k, \tag{3.9}$$

$$\#\{k : 1 \leqslant k \leqslant S,\ x \in B_k\} \leqslant C_1(d) \tag{3.10}$$

for any $y \in \mathbb{R}^d$ and

$$\varphi(B_k) \leqslant \frac{\mu(\mathbb{R}^d)}{m} \leqslant \varphi(\overline{B}_k), \quad k = 1, \dots, S. \tag{3.11}$$

Denote by E the orthogonal complement of the subspace $H^l(\widetilde{\Omega}_1)$ defined by $\int_{B_k} fp\,dx = 0$, $k = 1, \dots, S$, where $p \in \mathcal{P}^{l-1}$, (see Lemma 3.1). It follows from Corollary 3.5 and (3.11) that for any function $f \perp E$

$$\int_{B_k} |f|^2\,d\mu \leqslant C_2 \frac{\mu(\mathbb{R}^d)}{m} \int_{B_k} |\nabla^l f|^2\,dx.$$

Then using the last inequality, (3.9) and (3.10) we obtain the required statement.

Therefore in order to finish the proof of the lemma we need to construct a family of balls satisfying conditions (3.8)-(3.11).

From (3.5) and (3.7) it is easy to see that $\lim_{r\to 0} \varphi(B(x,r)) = 0$. By applying Lemma 2.2, where φ is used instead of μ, we find a family of balls $\{B_k\}_{k=1}^S$ such that (3.9)-(3.11) are fulfilled. We only need to check (3.8).

Choose $0 < \delta \leqslant (\varkappa/K)^{\frac{\beta}{\beta-\alpha}}$. Let us split the family $\{B_k\}_{k=1}^S$ into two sets of balls which after renumbering satisfy

$$\mu(\overline{B}_k) < \frac{\delta\,\mu(\mathbb{R}^d)}{m}, \qquad 1 \leqslant k \leqslant s, \tag{3.12}$$

and

$$\mu(\overline{B}_k) \geqslant \frac{\delta\,\mu(\mathbb{R}^d)}{m}, \qquad s+1 \leqslant k \leqslant S.$$

The condition (3.10) gives us $S - s \leqslant C_1(d)/\delta m$. Thus in order to complete the proof of (3.8) it is enough to verify the estimate

$$s \leqslant C_3\,m. \tag{3.13}$$

From now on we use the notations from Lemma 3.6. Let us claim that for any $B_k = B(x_k, r_k)$, $1 \leqslant k \leqslant s$, there is a ball $B_{u_k,i_k} = B(x_{u_k i_k}, r_{u_k i_k})$ with the properties

$$|x_k - x_{u_k i_k}| \leqslant 4r_k, \quad r_k/\zeta 2^u \leqslant r_{u_k i_k} \leqslant r_k/2^u.$$

Then from these inequalities and (3.10) we find that for any $0 \leqslant v \leqslant L-1$ and $1 \leqslant j \leqslant n_v$

$$\#\{k : 1 \leqslant k \leqslant s,\, u_k = v,\, i_k = j\} \leqslant C_4(d)C_1(d) = C_5.$$

Hence by (3.6)

$$s \leqslant C_5(n_0 + n_1 + \dots + n_{L-1}) \leqslant C_5 C(A,d)\,m \sum_{v=0}^{L-1} A^{-v} \leqslant C_6(A,d)m$$

and therefore (3.13) is proved.

Let us prove our claim. From (3.11) we conclude that for any $1 \leqslant k \leqslant s$ there exists a ball $B(y_k, \rho_k)$, such that $\rho_k \leqslant r_k$, $y_k \in \overline{B}_k$ and

$$\mu(\overline{B}(y_k, \rho_k) \cap \overline{B}(x_k, r_k))(r_k/\rho_k)^\beta \geqslant \frac{\mu(\mathbb{R}^d)}{m}. \tag{3.14}$$

The latter and (3.12) imply

$$\left(\frac{r_k}{\rho_k}\right)^\beta \delta > 1. \tag{3.15}$$

Using now (3.5), (3.14) and (3.15) we obtain

$$\begin{aligned}\mu(B(y_k, r_k)) &\geqslant \varkappa\left(\frac{r_k}{\rho_k}\right)^\alpha \mu(\overline{B}(y_k, \rho_k)) \\ &\geqslant \varkappa\left(\frac{r_k}{\rho_k}\right)^\alpha \mu(\overline{B}(y_k, \rho_k) \cap \overline{B}(x_k, r_k)) \\ &\geqslant \varkappa\left(\frac{r_k}{\rho_k}\right)^{\alpha-\beta} \frac{\mu(\mathbb{R}^d)}{m} \\ &\geqslant \varkappa\left(\frac{1}{\delta^{1/\beta}}\right)^{\alpha-\beta} \frac{\mu(\mathbb{R}^d)}{m} \geqslant \frac{K\,\mu(\mathbb{R}^d)}{m},\end{aligned}$$

where the last inequality follows from the choice of the constant δ. By applying Lemma 3.6 to $B(y_k, r_k)$ we find the required ball B_{u_k, i_k} and hence prove the claim and the lemma. □

From Lemma 3.4 and Condition $(*)$ also we obtain the following statement:

Lemma 3.8. *Let $l = d/2 \in \mathbb{N}$ and let μ be a nonnegative, finite measure satisfying Condition $(*)$ with the constants (γ_1, γ_2) and $\operatorname{supp}\mu \subset \overline{\Omega}_1$. Then*

$$\int_{\widetilde{\Omega}_1} |f(x)|^2\, d\mu(x) \leqslant C''\, \mu(\overline{\Omega}_1)\Big(\int_{\widetilde{\Omega}_1} |\nabla^l f(x)|^2\, dx + \int_{\widetilde{\Omega}_1} |f(x)|^2\, dx\Big),$$

where $C'' = C(d, \gamma_1, \gamma_2)$.

4. Proof of Theorem 1.1

According to the variational principle, in order to prove Theorem 1.1 it is sufficient to show that there exists a subspace $E_0 \subset H^l(\mathbb{R}^d)$, $\dim E_0 \leqslant C\,\mu(\mathbb{R}^d)$, such that for any $F \in \mathcal{H}^l$ and $f \perp E_0$ in $L^2(\mathbb{R}^d)$ we have the following inequality

$$\int_{\mathbb{R}^d} |f|^2\, d\mu \leqslant \int_{R^d} |\nabla^l f|^2\, dx + \int_{R^d} \frac{|f|^2}{|x|^{2l}}\, dx. \tag{4.1}$$

Let us denote by μ_k the restriction of μ on the set Ω_k. Introduce

$$\mathcal{K} := \Big\{k :\ \|\mu_k\|_1 > \frac{1}{3 \cdot 2^d \cdot C''}\Big\},$$

where C'' is defined in Lemma 3.8.

Lemma 3.7 implies that for any $k \in \mathcal{K}$ and any $m = m_k \in \mathbb{N}$ we can find a $C(d) \cdot m_k$ - dimensional subspace $E_k \subset L^2(\widetilde{\Omega}_k)$, such that for $f \perp E_k$ we have

$$\begin{aligned}(4.2)\quad \int_{\overline{\Omega}_k} |f(x)|^2\, d\mu(x) &= \int_{\overline{\Omega}_1} |f(2^k x)|^2\, d\mu(2^k x) \\ &\leqslant C' \,\frac{\mu(2^k\,\Omega_1)}{m_k} \int_{\widetilde{\Omega}_1} |\nabla^l f(2^k x)|^2\, dx \leqslant C' \,\frac{\mu(\Omega_k)}{m_k} \int_{\widetilde{\Omega}_k} |\nabla^l f(x)|^2\, dx.\end{aligned}$$

Notice that if we now choose $m_k = 3\,[(1 + C^{'}) \cdot \mu(\Omega_k)]$, then

$$(4.3)\qquad \int_{\overline{\Omega}_k} |f(x)|^2\, d\mu(x) \leqslant \frac{1}{3} \int_{\widetilde{\Omega}_k} |\nabla^l f(x)|^2\, dx,$$

and moreover

$$(4.4)\qquad \sum_{k\in\mathcal{K}} m_k \leqslant 3(1 + C^{'}) \cdot \mu(\mathbb{R}^d).$$

Assume now that $k \notin \mathcal{K}$. Then Lemma 3.8 and the definition of the set $\mathcal{K}$ give us

$$\begin{aligned}(4.5)\quad \int_{\overline{\Omega}_k} |f(x)|^2\, d\mu(x) &= \int_{\overline{\Omega}_1} |f(2^k x)|^2\, d\mu(2^k x) \\ &\leqslant C^{''}\, \mu(2^k\,\Omega_1) \Big(\int_{\widetilde{\Omega}_1} |\nabla^l f(2^k x))|^2\, dx + \int_{\widetilde{\Omega}_1} |f(2^k x)|^2\, dx \Big) \\ &= C^{''}\, \mu(\Omega_k) \Big(\int_{\widetilde{\Omega}_k} |\nabla^l f(x)|^2\, dx + 2^{-dk} \int_{\widetilde{\Omega}_k} |f(x)|^2\, dx \Big) \\ &\leqslant 2^d C^{''}\, \mu(\Omega_k) \Big(\int_{\widetilde{\Omega}_k} |\nabla^l f(x)|^2\, dx + \int_{\widetilde{\Omega}_k} \frac{|f(x)|^2}{|x|^{2l}}\, dx \Big).\end{aligned}$$

This inequality and the definition of $\mathcal{K}$ imply

$$(4.6)\qquad \int_{\overline{\Omega}_k} |f(x)|^2\, d\mu(x) \leqslant \frac{1}{3} \Big(\int_{\widetilde{\Omega}_k} |\nabla^l f(x)|^2\, dx + \int_{\widetilde{\Omega}_k} \frac{|f(x)|^2}{|x|^{2l}}\, dx \Big).$$

Summing up the inequalities (4.3) and (4.6) we obtain (4.1). Besides, (4.4) gives $\dim E_0 = \sum_{k\in\mathcal{K}} \dim E_k = \sum_{k\in\mathcal{K}} m_k \leqslant C(d)\mu(\mathbb{R}^d)$. The theorem is proved. □

5. Proof of Theorem 1.2

5.1. Some properties of L^p classes of functions. Let $Q = (0, 1)^d$, $d \in \mathbb{N}$. We begin with an auxiliary statement.

Proposition 5.1. *Let $f \geqslant 0$ and $f \in L^p(Q)$, $1 < p \leqslant \infty$. Then there exists $g \in L^p(Q)$, such that $g \geqslant f$ a.e.,*

$$\|g\|_{L^p(Q)} \leqslant C(p, d)\|f\|_{L^p(Q)}$$

and the measure $g\,dx$ satisfies Condition $(*)$ with some constants $\gamma_1 = \gamma_1(p, d)$ and $\gamma_2 = \gamma_2(p, d)$.

Proof. Let $u \in L^p(\mathbb{R}^d)$ and let $Pu = \chi_Q u = u|_Q$ be the restriction of u to the cube Q. Introduce the Hardy-Littlewood maximal function

$$\mathcal{M} f(x) = \sup_{\rho>0} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y)|\, dy.$$

Then by using the Hardy-Littlewood-Wiener theorem (see for example Th.I.1 in [St]) we find that there is a constant $A = A(p)$, such that

$$\|P\mathcal{M} f\|_p \leqslant \|\mathcal{M} f\|_p \leqslant A\|f\|_{L^p(Q)}.$$

Define (cf. [GR])

$$g(x) = \sum_{k=0}^{\infty} 2^{-k} A^{-k} (P\mathcal{M})^k f(x). \tag{5.1}$$

Obviously $\operatorname{supp} g \subset Q$, $f \leqslant g$ a.e., $\|g\|_p \leqslant 2\,\|f\|_p$, and

$$P\mathcal{M}\, g(x) \leqslant 2A\, g(x). \tag{5.2}$$

It only remains to check that the measure $g\,dx$ satisfies Condition $(*)$. Thus we should find constants (γ_1, γ_2), such that for any $x_0 \in \mathbb{R}^d$ and $r \leqslant \gamma_2^{-1}\sqrt{d}$

$$\int_{B(x_0,\gamma_1 r)} g(x)\, dx \geqslant 2 \int_{B(x_0,r)} g(x)\, dx. \tag{5.3}$$

Let $\gamma_1 = \gamma_2 = \gamma > 1$ be a constant whose value is to be found. Then for any $x \in Q \cap \{B(x_0,\gamma r) \setminus B(x_0,r)\}$ the inequality (5.2) implies

$$g(x) \geqslant \frac{1}{2A\,|B(x, r+|x-x_0|)|} \int_{B(x,r+|x-x_0|)} g(y)\, dy \\ \geqslant \frac{1}{2Av_d\,(r+|x-x_0|)^d} \int_{B(x_0,r)} g(y)\, dy.$$

Integrating this inequality over the set $Q \cap \{B(x_0,\gamma r) \setminus B(x_0,r)\}$ we obtain

$$\begin{aligned}
&\int_{Q\cap\{B(x_0,\gamma r)\setminus B(x_0,r)\}} g(x)\, dx \\
&\quad\geqslant \frac{1}{2Av_d} \int_{Q\cap\{B(x_0,\gamma r)\setminus B(x_0,r)\}} \frac{1}{(r+|x-x_0|)^d}\, dx \int_{B(x_0,r)} g(y)\, dy \\
&\quad\geqslant \frac{1}{2^{d+1}Av_d} \int_{B(x_0,\gamma r)\setminus B(x_0,r)} \frac{1}{(r+|x-x_0|)^d}\, dx \int_{B(x_0,r)} g(y)\, dy \\
&\quad= \frac{v_d d}{2^{d+1}Av_d} \int_r^{\gamma r} \frac{u^{d-1}}{(r+u)^d}\, du \int_{B(x_0,r)} g(y)\, dy \\
&\quad\geqslant \frac{d}{2^{d+1}A} \int_r^{\gamma r} \frac{u^{d-1}}{(u+u)^d}\, du \int_{B(x_0,r)} g(y)\, dy = \frac{d}{2^{2d+1}A} \ln\gamma \int_{B(x_0,r)} g(y)\, dy.
\end{aligned}$$

If we now choose γ, such that

$$d \ln \gamma \geqslant 2^{2d+2} A,$$

then (5.3) is satisfied and therefore the proof is complete. □

When defining Condition $(*)$ we used the family of balls. There is a natural question whether instead of balls we can use other families of sets. The next lemma answers this question.

Let $0 \in G \subset \mathbb{R}^d$ be a domain, such that

$$B(0, r_1) \subset G \subset B(0, r_2) \tag{5.4}$$

with some $0 < r_1 < r_2 < \infty$. Define

$$G(x, r) = \{y \in \mathbb{R}^d : (y - x)/r \in G, \}, \qquad r > 0. \tag{5.5}$$

Lemma 5.2. *Let G be a set satisfying (5.4) and let μ be a non-negative measure, $F = \operatorname{supp} \mu$. The following two properties are equivalent:*

- (i) *There exist constants γ_1 and γ_2, such that Condition $(*)$ holds with constants (γ_1, γ_2).*
- (ii) *There exist constants γ_1' and γ_2', such that for any $x \in \mathbb{R}^d$ and $r \leqslant \operatorname{diam} F/\gamma_2'$ we have*

$$\mu(G(x, \gamma_1' r)) \geqslant 2\, \mu(G(x, r)). \tag{5.6}$$

Proof. Suppose (i) is satisfied. Let us check (ii). For any $x \in \operatorname{supp} \mathbb{R}^d$ and $r \leqslant \frac{\operatorname{diam} F}{\gamma_2 r_2}$ we have that

$$\mu(G(x, \gamma_1 r_2 r/r_1)) \geqslant \mu(B(x, g_1 r_2\, r) \geqslant 2\mu(B(x, r_2 r)) \geqslant 2\mu(G(x, r)).$$

The latter implies $\gamma_1' = \gamma_1 r_2/r_1$ and $\gamma_2' = r_2 \gamma_2$. The converse statement can be proved analogously. □

In the proof of the next statement it is convenient to use a family of cubes

$$Q(x, r) = \{y \in \mathbb{R}^d : |y - x|/r \in (-1, 1)^d\},$$
$$Q_1(x_1, r) = \{y_1 \in \mathbb{R}^{d_1} : |y_1 - x_1|/r \in (-1, 1)^{d_1}\},$$
$$Q_2(x_2, r) = \{y_2 \in \mathbb{R}^{d_2} : |y_2 - x_2|/r \in (-1, 1)^{d_2}\}.$$

Proposition 5.3. *Let $Q = Q_1 \times Q_2 = (0, 1)^{d_1} \times (0, 1)^{d_2}$, $d = d_1 + d_2$, $f \geqslant 0$ and $f \in L^1(Q_1, L^p(Q_2))$, $1 < p \leqslant \infty$. Then there exists $g \in L^1(Q_1, L^p(Q_2))$, such that $g \geqslant f$ a.e.,*

$$\|g\|_{L^1(Q_1, L^p(Q_2))} \leqslant C(p, d_1, d_2) \|f\|_{L^1(Q_1, L^p(Q_2))}$$

and the measure $g\, dx$ satisfies Condition $(*)$ with the constants $\gamma_1 = \gamma_1(p, d_1, d_2)$ and $\gamma_2 = \gamma_2(p, d_1, d_2)$.

Proof. For the functions $f(x_1, \cdot) \in L^p(Q_2)$, $x_1 \in Q_1$, we introduce $g(x_1, \cdot) \in L^p(Q_2)$ according the construction in Proposition 5.1. Clearly $g(x_1, x_2)\, dx_2$ satisfies Condition $(*)$ for the family of cubes $Q_2(x_2, r)$, $x_2 \in \mathbb{R}^{d_2}$, with constants (γ_1, γ_2) uniformly with respect to $x_1 \in Q_1$. In order to check Condition $(*)$ for the function g

we prove (5.6) for the family of cubes $Q(x,r)$, $x=(x_1,x_2)$. Indeed, for any $x\in\mathbb{R}^d$ and $r<\sqrt{d}/\gamma_2$ we have

$$\int_{Q(x,\gamma_1 r)} g(y)\,dy = \int_{Q_1(x,\gamma_1 r)}\int_{Q_2(x,\gamma_1 r)} g(y_1,y_2)\,dy_2dy_1 \geqslant 2\int_{Q_1(x,\gamma_1 r)}\int_{Q_2(x,r)} g(y_1,y_2)\,dy_2dy_1 \geqslant 2\int_{Q_1(x,r)}\int_{Q_2(x,r)} g(y_1,y_2)\,dy_2dy_1.$$

The proposition is proved. □

Corollary 5.4. *The statement of Proposition 5.3 holds true if we replace the cube Q_2 by $\mathbb{S}^{d_2}$.*

5.2. Proof of Theorem 1.2. In the polar coordinates $x=(r,\theta)\in\mathbb{R}^+\times\mathbb{S}^{d-1}$, every set Ω_k turns into $[2^k,2^{k+1})\times\mathbb{S}^{d-1}$. According to Proposition 5.3 we find functions $g_k\in L^1((2^k,2^{k+1}),L^p(\mathbb{S}^{d-1}))$, such that $g:=\sum_k g_k\geqslant V$ a.e., $\|g\|_{L^1(\mathbb{R}_+,L^p(\mathbb{S}^{d-1}))}\leqslant C(p,d)\|V\|_{L^1(\mathbb{R}_+,L^p(\mathbb{S}^{d-1}))}$ and the measures $g_k\,dx$ satisfy Condition $(*)$ with constants (γ_1,γ_2) which are independent of k. Finally we have

$$N(V)\leqslant N(g)\leqslant C_1\int g(x)\,dx\leqslant C_2\|g\|_{L^1(\mathbb{R}_+,L^p(\mathbb{S}^{d-1}))}\leqslant C_3\,\|V\|_{L^1(\mathbb{R}_+,L^p(\mathbb{S}^{d-1}))},$$

where $C_j=C_j(d,p)$, $j=1,2,3$. This completes the proof. □

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