ONE-DIMENSIONAL INTERPOLATION INEQUALITIES, CARLSON–LANDAU INEQUALITIES AND MAGNETIC SCHRÖDINGER OPERATORS

ALEXEI ILYIN, ARI LAPTEV, MICHAEL LOSS, SERGEY ZELIK

Abstract. In this paper we prove refined first-order interpolation inequalities for periodic functions and give applications to various refinements of the Carlson–Landau-type inequalities and to magnetic Schrödinger operators. We also obtain Lieb-Thirring inequalities for magnetic Schrödinger operators on multi-dimensional cylinders.

1. Introduction

The Carlson inequality [3]

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 \leqslant \pi\left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2}\left(\sum_{k=1}^{\infty} k^2 a_k^2\right)^{1/2} \tag{1.1}$$

has been a source of many improvements, refinements and generalizations (see [7],[12] and the references therein). The constant π here is sharp and the inequality is strict unless $\{a_k\}_{k=1}^{\infty} \equiv 0$.

This inequality and its various generalizations are closely connected with classical one-dimensional interpolation inequalities for Sobolev spaces:

$$\|u\|_\infty^2 \leq \mathrm{C}(m)\|u\|^{2\theta}\|u^{(m)}\|^{2(1-\theta)}, \quad \theta = 1 - \frac{1}{2m},\ m > \frac{1}{2}. \tag{1.2}$$

In the case when $x \in \mathbb{R}$ the sharp constant and the corresponding extremals were found in [13]:

$$\mathrm{C}(m) = \frac{1}{\theta^\theta(1-\theta)^{1-\theta}2m\sin\frac{\pi}{2m}}.$$

In the periodic case $x \in (0, 2\pi)$ with zero average condition the inequality holds with the same constant (without extremal functions) [9].

2000 *Mathematics Subject Classification.* 26D10, 35P15, 46E35.

Key words and phrases. Sobolev inequality, interpolation inequalities, Carlson inequality, Lieb–Thirring inequalities.

Furthermore, the first-order inequality with C(1) = 1 is equivalent (as was first observed in [6]) to (1.1) by going over from $\{a_k\}_{k=1}^\infty$ to $u(x) = \sum_{k\in\mathbb{Z}_0} a_{|k|}e^{ikx}$ and using Parseval's equality. Here in what follows $\|\cdot\|_\infty := \|\cdot\|_{L_\infty}$, $\|\cdot\| := \|\cdot\|_{L_2}$, and $\mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$.

For all $m > 1/2$ inequality admits a negative correction term on the right-hand side [2], [14], in particular, in the first- and second-order cases the correction term can be written in closed form

$$\|u\|_\infty^2 \le \|u\|\,\|u'\| - \frac{1}{\pi}\|u\|^2, \tag{1.3}$$

$$\|u\|_\infty^2 \le \frac{\sqrt{2}}{\sqrt[4]{27}}|u\|\,\|u'\| - \frac{2}{3\pi}\|u\|^2, \tag{1.4}$$

where all constants are sharp and no extremals exist. Again, for $u(x) = \sum_{k\in\mathbb{Z}_0} a_{|k|}e^{ikx}$ this gives the following two improved Carlson inequalities

$$\left(\sum_{k=1}^\infty a_k\right)^2 \le \pi\left(\sum_{k=1}^\infty a_k^2\right)^{1/2}\left(\sum_{k=1}^\infty k^2a_k^2\right)^{1/2} - \sum_{k=1}^\infty a_k^2, \tag{1.5}$$

$$\left(\sum_{k=1}^\infty a_k\right)^2 \le \frac{\sqrt{2}\pi}{\sqrt[4]{27}}\left(\sum_{k=1}^\infty a_k^2\right)^{3/4}\left(\sum_{k=1}^\infty k^4a_k^2\right)^{1/4} - \frac{2}{3}\sum_{k=1}^\infty a_k^2, \tag{1.6}$$

with sharp constants. These inequalities are proved in [2],[14] in the framework of a rather general theory and we give below in § 2 a new direct self-contained proof of (1.3) and (1.5).

In § 3 we consider inequalities of the form

$$\|u\|_\infty^2 \le K(\alpha)\|A^{1/2}u\|\|u\| \tag{1.7}$$

for 2π-periodic functions (no zero average condition), where

$$\|A^{1/2}u\|^2 = \int_0^{2\pi}\left|i\,\frac{du}{dx} - a(x)u\right|^2 dx$$

is the quadratic form corresponding to the Schrödinger operator

$$Au = \left(i\,\frac{du}{dx} - a(x)u\right)^2$$

with magnetic potential $a \in L_1(0, 2\pi)$. The sharp constant $K(\alpha)$ depends only on the flux

$$\alpha := \frac{1}{2\pi}\int_0^{2\pi} a(x)dx,$$

it is finite if and only if $\alpha \notin \mathbb{Z}$. In a somewhat similar situation considered in [10] the introduction of a magnetic field has made it possible to prove the Hardy inequality in $\mathbb{R}^2$. In our periodic case a magnetic field with non-integral flux removes the condition $\int_0^{2\pi} u(x)dx = 0$.

The expression for $K(\alpha)$ is as follows

$$K(\alpha) = \begin{cases} |\sin(2\pi\alpha)|^{-1}, & \alpha \bmod(1) \in (0, 1/4) \cup (3/4, 1); \\ 1, & \alpha \bmod(1) \in [1/4, 3/4]. \end{cases} \tag{1.8}$$

In the first case there exists a unique extremal function and for $\alpha \in [1/4, 3/4]$ there are no extremals and a negative correction term may exist. We show in § 4 that this is indeed the case and

$$\begin{aligned} &\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|\bigl(1 - 2e^{-4\pi\|A^{1/2}u\|/\|u\|}\bigr),\ \alpha = 1/4,\ \alpha = 3/4, \\ &\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|\bigl(1 + 2\cos(2\pi\alpha)e^{-2\pi\|A^{1/2}u\|/\|u\|}\bigr),\ \alpha \in (1/4, 3/4). \end{aligned} \tag{1.9}$$

In § 5 we consider applications to Carlson–Landau inequalities. The Landau improvement of (1.1) (see, for instance, [7])

$$\left(\sum_{k=1}^\infty a_k\right)^2 \leqslant \pi \left(\sum_{k=1}^\infty a_k^2\right)^{1/2} \left(\sum_{k=1}^\infty (k-1/2)^2 a_k^2\right)^{1/2} \tag{1.10}$$

has a surprisingly short (and almost elementary) proof in terms of our interpolation inequalities. We recall the elementary inequality (which is (1.2) with $m = 1$)

$$\|u\|_\infty^2 \le \|u\|\|u'\|, \qquad u \in H_0^1(0, L), \tag{1.11}$$

following from

$$2u(x)^2 = \int_0^x (u(t)^2)'dt - \int_x^L (u(t)^2)'dt \le 2\|u\|\|u'\|.$$

Given a (non-negative) sequence $\{a_k\}_{k=1}^\infty$ we set $L = 1$ and consider the function

$$u(x) = \sqrt{2}\sum_{k=1}^\infty (-1)^{k+1} a_k \sin(2k-1)\pi x, \quad x \in [0, 1]. \tag{1.12}$$

We have $\|u\|_\infty = u(\pi/2) = \sqrt{2}\sum_{k=1}^\infty a_k$ and by orthonormality,

$$\|u\|^2 = \sum_{k=1}^\infty a_k^2, \qquad \|u'\|^2 = \pi^2 \sum_{k=1}^\infty (2k-1)^2 a_k^2 = 4\pi^2 \sum_{k=1}^\infty (k-1/2)^2 a_k^2. \tag{1.13}$$

Substituting this into (1.11) we obtain inequality (1.10).

The refinement of (1.11) obtained in [14]

$$\|u\|_\infty^2 \le \|u\|\|u'\|\bigl(1-2e^{-L\|u'\|/\|u\|}\bigr), \qquad u\in H_0^1(0,L) \tag{1.14}$$

or, equivalently, inequality (1.9) in the symmetric case $\alpha=1/2$ give a sharp correction term to the Carlson–Landau inequality (1.10)

$$\biggl(\sum_{k=1}^\infty a_k\biggr)^2 \le \pi\|a\|\|a\|_1\bigl(1-2e^{-2\pi\|a\|_1/\|a\|}\bigr). \tag{1.15}$$

Next, using a second-order inequality in [14] we obtain the following sharp inequality

$$\biggl(\sum_{k=1}^\infty a_k\biggr)^2 \le \frac{\sqrt{2}\,\pi}{\sqrt[4]{27}}\coth(\pi/2)\|a\|^{3/2}\|a\|_2^{1/2}$$

with unique extremal $a_k=1/(2k-1)^4+4)$. Here we set for brevity

$$\|a\|^2=\sum_{k=1}^\infty a_k^2,\quad \|a\|_1^2=\sum_{k=1}^\infty(k-1/2)^2a_k^2,\quad \|a\|_2^2=\sum_{k=1}^\infty(k-1/2)^4a_k^2. \tag{1.16}$$

The whole family of Carlson–Landau inequalities

$$\biggl(\sum_{k=1}^\infty a_k\biggr)^2 \le k(\alpha)\biggl(\sum_{k=1}^\infty a_k^2\biggr)^{1/2}\biggl(\sum_{k=1}^\infty(k-\alpha)^2a_k^2\biggr)^{1/2}, \tag{1.17}$$

is studied for $\alpha\in[0,1)$ in Theorem 5.2. Obviously, $k(\alpha)=\pi$ for $\alpha\in[0,1/2]$ and, furthermore, for $\alpha\in[0,1/2)$ we have a sharp L_2-type correction term here, see (5.6). In the symmetric case $\alpha=1/2$ the correction term is exponentially small, see (1.15). For $\alpha\in(1/2,1)$ we show that $k(\alpha)>\pi$, moreover, $k(\alpha)\sim(1-\alpha)^{-1}$ as $\alpha\to1^-$, and there exists a unique extremal.

Finally, in § 6 we consider applications to the Lieb–Thirring inequalities and first give a new alternative proof of the main result in [4] on the one-dimensional Sobolev inequalities for orthonormal families of vector-functions along with generalizations to higher-order derivatives and 1-D magnetic forms. This gives the Lieb–Thirring estimate for the negative trace of a 1-D magnetic Schrödinger operator with a matrix-valued potential. Then we combine this result with the main ideas and results in [1], [4], [8], and [11] to obtain in Theorem 6.4 estimates for the 1/2- and 1- moments of the negative eigenvalues of the

Schrödinger-type operator in $\mathbb{T}_x^{d_1} \times \mathbb{R}_y^{d_2}$. For example, for $d_1 = d_2 = 1$ and the operator

$$\mathcal{H}\Psi = -\frac{d^2}{dy^2}\Psi + \left(i\,\frac{d}{dx} - a(x)\right)^2 \Psi - V(x,y)\Psi = -\lambda\Psi$$

on the cylinder $\mathbb{R}_y \times \mathbb{S}_x^1$ we have the following estimates for its negative eigenvalues:

$$\begin{aligned}\sum_k \lambda_k^{1/2} &\le \frac{1}{3\sqrt{3}} K(\alpha) \int_{\mathbb{R}\times\mathbb{S}^1} V^{3/2}(x,y)dydx, \\ \sum_k \lambda_k &\le \frac{1}{8\sqrt{3}} K(\alpha) \int_{\mathbb{R}\times\mathbb{S}^1} V^{2}(x,y)dydx.\end{aligned} \tag{1.18}$$

For $d_1 = 2$, $d_2 = 0$ and the operator

$$\mathcal{H}\Psi = \left(i\,\frac{d}{dx_1} - a_1(x_1)\right)^2 \Psi + \left(i\,\frac{d}{dx_2} - a_2(x_2)\right)^2 \Psi - V(x,y)\Psi = -\lambda\Psi$$

on the torus $\mathbb{T}^2$ with $\alpha_j = \frac{1}{2\pi}\int_0^{2\pi} a(x_j)dx \notin \mathbb{Z}$, $j = 1,2$ we have

$$\sum_k \lambda_k \le \frac{\pi}{24} K(\alpha_1)K(\alpha_2) \int_{\mathbb{T}^2} V^2(x_1,x_2)dx_1dx_2. \tag{1.19}$$

Note that in the region where $K(\alpha) = 1$, the constants in (1.18) coincide with the best-known constants in the corresponding Lieb–Thirring inequalities for the Schrodinger operator in $\mathbb{R}^2$, see [4], [8]. However, the constant in (1.19) contains an extra factor $\pi/\sqrt{3}$, since when we apply "the lifting argument with respect to dimensions" [11] in the direction x, we do not have semiclassical estimates for the γ-Riesz means with $\gamma \ge 3/2$ for the negative eigenvalues in the periodic case. This factor along with $K(\alpha_j)$ accumulates with each iteration of the lifting procedure with respect to the x-variables, see Theorem 6.4.

2. Proof of first-order inequality

We consider the following maximization problem for 2π-periodic functions with zero average: for $D \ge 1$ find $\mathbb{V}(D)$ – the solution of the following extremal problem

$$\mathbb{V}(D) := \sup\bigl\{|u(0)|^2,\ \|u\|^2 = 1,\ \|u'\|^2 = D\bigr\}. \tag{2.20}$$

The next lemma gives an implicit formula for the function $\mathbb{V}(D)$.

Lemma 2.1. *The following expression holds for* $\mathbb{V}(D)$*:*

$$\mathbb{V}(D)=\frac{1}{2\pi}\frac{\left(\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2}\right)^2}{\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2}}\,, \tag{2.21}$$

where $\lambda=\lambda(D)$ *is a unique solution of the functional equation*

$$\frac{\sum_{k\in\mathbb{Z}_0}\frac{k^2}{(\lambda+k^2)^2}}{\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2}}=D\ (=:D(\lambda)). \tag{2.22}$$

Furthermore, $\lambda(1)=-1$ *and* $\lambda(\infty)=\infty$*.*

Proof. Using the Fourier series $u(x)=\sum_{k\in\mathbb{Z}_0}u_ke^{ikx}$ and the Parseval equalities $\|u\|^2=2\pi\sum_{k\in\mathbb{Z}_0}|u_k|^2$, $\|u'\|^2=2\pi\sum_{k\in\mathbb{Z}_0}k^2|u_k|^2$, for every $\lambda>-1$ we have by the Cauchy–Schwartz inequality

$$\begin{aligned}|u(0)|^2=\biggl|\sum_{k\in\mathbb{Z}_0}u_k\biggr|^2\le\biggl(\sum_{k\in\mathbb{Z}_0}|u_k|\biggr)^2=\\=\biggl(\sum_{k\in\mathbb{Z}_0}|u_k|(\lambda+k^2)^{1/2}(\lambda+k^2)^{-1/2}\biggr)^2\le\\\le\sum_{k\in\mathbb{Z}_0}|u_k|^2(\lambda+k^2)\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2}=\\=\frac{1}{2\pi}(\lambda\|u\|^2+\|u'\|^2)\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2}=\\=\frac{1}{2\pi}\|u\|^2(\lambda+\|u'\|^2/\|u\|^2)\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2}\,.\end{aligned} \tag{2.23}$$

Moreover, for (and only for)

$$u_k=u_k^{(\lambda)}:=\mathrm{const}\frac{1}{\lambda+k^2}$$

the above inequalities turn into equalities. Fixing for definiteness const $:=\frac{1}{2\pi}$ we consider the function $G_\lambda(x)=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0}\frac{e^{ikx}}{\lambda+k^2}$, for which

$$\frac{\|(G_\lambda)'_x\|^2}{\|G_\lambda\|^2}=\frac{\sum_{k\in\mathbb{Z}_0}\frac{k^2}{(\lambda+k^2)^2}}{\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2}}=:D(\lambda).$$

(We also observe that G_λ solves the equation $-\frac{d^2}{dx^2}G_\lambda+\lambda G_\lambda=\delta$.)

Then we see that for every fixed $\lambda > -1$ the normalized function $G_\lambda(x)/\|G_\lambda\|$ is the extremal function in problem (2.20) with $D = D(\lambda)$, and the value of the solution function $\mathbb{V}(D)$ is

$$\mathbb{V}(D) = \frac{1}{2\pi}(\lambda + D(\lambda)) \sum_{k\in\mathbb{Z}_0} \frac{1}{\lambda+k^2} =$$

$$= \frac{1}{2\pi}\left(\lambda + \frac{\sum_{k\in\mathbb{Z}_0} \frac{k^2}{(\lambda+k^2)^2}}{\sum_{k\in\mathbb{Z}_0} \frac{1}{(\lambda+k^2)^2}}\right) \sum_{k\in\mathbb{Z}_0} \frac{1}{\lambda+k^2} = \frac{1}{2\pi}\,\frac{(\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2})^2}{\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2}}\,.$$

We now have to show that there exists a unique $\lambda = \lambda(D)$ solving (2.22) for every fixed $D \ge 1$. We first observe that $D(\lambda) \to 1$ as $\lambda \to -1$ and $D(\lambda) \to \infty$ as $\lambda \to \infty$. It remains to show that $D(\lambda)$ is strictly monotone increasing. We have

$$\frac{d}{d\lambda}D(\lambda) = \frac{2}{(\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2})^2} \cdot \sum_{k,l\in\mathbb{Z}_0} \frac{k^2(k^2-l^2)}{(\lambda+k^2)^3(\lambda+l^2)^3} =$$

$$= \frac{2}{(\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2})^2} \cdot \sum_{k,l\in\mathbb{Z}_0, k>l} \frac{k^2(k^2-l^2)+l^2(l^2-k^2)}{(\lambda+k^2)^3(\lambda+l^2)^3} =$$

$$= \frac{2}{(\sum_{k\in\mathbb{Z}_0}\frac{1}{(\lambda+k^2)^2})^2} \cdot \sum_{k,l\in\mathbb{Z}_0, k>l} \frac{(k^2-l^2)^2}{(\lambda+k^2)^3(\lambda+l^2)^3} \;>\; 0.$$

The proof is complete. □

We set

$$G(\lambda) := G_\lambda(0) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2}\,. \tag{2.24}$$

The following variational characterization of $\mathbb{V}(D)$ is important.

Theorem 2.1. *For a fixed $D \ge 1$*

$$\mathbb{V}(D) = \min_{\lambda\in[-1,\infty)} (\lambda + D)G(\lambda). \tag{2.25}$$

Proof. Since $G(\lambda) \to +\infty$ as $\lambda \to -1$ and $G(\lambda) = O(\lambda^{-1/2})$ as $\lambda \to +\infty$, it follows that the minimum is attained for each fixed $D \ge 1$ at some point $\lambda_* = \lambda_*(D)$. Then

$$\frac{d}{d\lambda}\big((\lambda + D)G(\lambda)\big)|_{\lambda=\lambda_*} = 0,$$

which gives

$$D = -\frac{G(\lambda)}{G'(\lambda)} - \lambda.$$

In view of (2.24) this equation coincides with (2.22) and therefore $\lambda_*(D)$ coincides with the unique inverse function $\lambda(D)$ constructed in Lemma 2.1. This gives

$$(\lambda(D)+D)G(\lambda(D)) = -\frac{G(\lambda(D))^2}{G'(\lambda(D))} = \mathbb{V}(D).$$

The proof is complete. □

Of course, it is impossible to find an explicit formula for the inverse function $\lambda = \lambda(D)$, therefore it is impossible to find an explicit formula for $\mathbb{V}(D)$. However, it is possible to find the asymptotic expansion of $\mathbb{V}(D)$ as $D \to \infty$. All that we need to know for this purpose is the asymptotic expansion of the function $G(\lambda)$ as $\lambda \to \infty$. This expansion, in turn, is found by the Poisson summation formula (or by means of the explicit formula (2.32)):

$$G(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2} = \frac{1}{2}\lambda^{-1/2} - \frac{1}{2\pi}\lambda^{-1} + O(e^{-\pi\lambda^{1/2}}) \quad \text{as } \lambda\to\infty. \tag{2.26}$$

Lemma 2.2. *It holds as* $D \to \infty$

$$\mathbb{V}(D) = D^{1/2} - \frac{1}{\pi} - \frac{1}{2\pi^2}D^{-1/2} + O(D^{-1}). \tag{2.27}$$

Proof. This is a particular case of the general result of Proposition 2.1 in [14]. In addition to (2.26) we have

$$G'(\lambda) = -\frac{1}{4}\lambda^{-3/2} + \frac{1}{2\pi}\lambda^{-2} + O(\lambda^{-5/2}), \tag{2.28}$$

and, hence,

$$D(\lambda) = -\frac{G(\lambda)}{G'(\lambda)} - \lambda = \lambda + \frac{2}{\pi}\lambda^{1/2} + \frac{4}{\pi^2} + O(\lambda^{-1/2}). \tag{2.29}$$

The well-defined inverse function $\lambda(D)$ (see (2.22)) has the asymptotic behaviour

$$\lambda(D) = D - \frac{2}{\pi}D^{1/2} - \frac{2}{\pi^2} + O(D^{-1/2}) \quad \text{as } D\to\infty. \tag{2.30}$$

Substituting this into (2.26), (2.28), we obtain for $\mathbb{V}(D) = -\frac{G^2(\lambda(D))}{G'(\lambda(D))}$ the asymptotic expansion (2.27). The proof is complete. □

The third term in (2.27) is negative, hence,

$$\mathbb{V}(D) < D^{1/2} - \frac{1}{\pi} \tag{2.31}$$

for all sufficiently large $D \geq D_0$. Therefore we shall have proved inequality (1.3) once we have shown that (2.31) holds for *all* $D \geq 1$. Moreover, Lemma 2.2 implies that both constants in (1.3) are sharp.

Theorem 2.2. *Inequality* (2.31) *holds for all* $D \geq 1$.

Corollary 2.1. *Inequalities* (1.3) *and* (1.5) *hold and all the constants there are sharp.*

Proof. By homogeneity and (2.31), for a $u \in \dot{H}^1$

$$u(0)^2 \leq \|u\|^2 \mathbb{V}\left(\frac{\|u'\|^2}{\|u\|^2}\right) < \|u\|^2 \cdot \frac{\|u'\|}{\|u\|} - \frac{1}{\pi}\|u\|^2.$$

□

Proof of the theorem. The proof is based on the variational representation (2.25) and the explicit formula for $G(\lambda)$:

$$G(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}_0}\frac{1}{\lambda+k^2} = \frac{1}{2\pi}\,\frac{\pi\sqrt{\lambda}\,\coth(\pi\sqrt{\lambda}\,) - 1}{\lambda}. \tag{2.32}$$

We estimate $G(\lambda)$ by a more convenient expression

$$G(\lambda) < \frac{\pi\sqrt{\lambda} - 1 + e^{-\pi\sqrt{\lambda}}}{2\pi\lambda} =: G_0(\lambda), \tag{2.33}$$

where the above inequality by equivalent transformations reduces to $x < \sinh(x)$, $x > 0$.

Thus, in view of (2.25) and (2.33), for $D \geq 1$

$$\mathbb{V}(D) \leq (\lambda + D)G_0(\lambda)\big|_{\lambda=(D^{1/2}-1/2)^2} =: \mathbb{V}_0(y(D)),$$

where $y = y(D) := D^{1/2} - 1/2$, $y \geq 1/2$ and

$$\mathbb{V}_0(y) = \frac{1}{2\pi y^2}\big(\pi y - 1 + e^{-\pi y}\big)\big(y^2 + (y+1/2)^2\big).$$

Now

$$\begin{gathered}\mathbb{V}(D) - D^{1/2} + \frac{1}{\pi} < \mathbb{V}_0(y) - \left(y + \frac{1}{2}\right) + \frac{1}{\pi} = \\ = \frac{1}{8\pi y^2}\Big((8y^2 + 4y + 1)e^{-\pi y} - (4-\pi)y - 1\Big) =: \frac{1}{8\pi y^2}W(y).\end{gathered} \tag{2.34}$$

Next,

$$W'(y) = \big(-8\pi y^2 + (16 - 4\pi)y + 4 - \pi\big)e^{-\pi y} - 4 + \pi$$

and the coefficient of $e^{-\pi y}$ is negative for $y \ge 1/2$. Therefore $W'(y) < 0$ and $W(y)$ is decreasing for $y \ge 1/2$ and

$$W(y) \le W(1/2) = 5e^{-\pi/2} - 3 + \pi/2 = -0.3898 < 0, \tag{2.35}$$

which completes the proof of (2.31). □

Remark 2.1. The proof of inequality (2.31) in the last theorem is in the spirit of Hardy's first proof [6] of the original Carlson inequality (1.1) and is, in fact, self-contained and formally independent of the previous argument. It follows from (2.23) that

$$\mathbb{V}(D) \le (\lambda + D)G(\lambda),$$

where $\lambda \ge -1$ is an arbitrary free parameter. Therefore inequality (2.31) will be proved if we succeed in finding such a substitution $\lambda = \lambda(D)$ for which

$$(\lambda(D) + D)G(\lambda(D)) < D^{1/2} - \frac{1}{\pi} \quad \text{for all } D \ge 1.$$

Now estimates (2.34) and (2.35) in the proof of Theorem 2.2 are saying that the substitution $\lambda(D) = D - D^{1/2} + 1/4$ will do the job. This substitution agrees in the leading term with (2.30). The lower order terms are 'experimental'. Also, without knowing (2.30) finding this substitution becomes much more difficult.

On the other hand, the proof of sharpness is contained in Lemma 2.2. Alternatively, we can verify sharpness of (1.3) (and (1.5)) at the test function $\sum_{k\in\mathbb{Z}_0} \frac{e^{ikx}}{\lambda+k^2}$ by letting $\lambda \to \infty$.

3. Magnetic inequality

We are interested in the inequality

$$\|u\|_\infty^2 \le K(\alpha) \left(\int_0^{2\pi} \left| i\,\frac{du}{dx} - a(x)u \right|^2 dx \right)^{1/2} \left(\int_0^{2\pi} |u(x)|^2 dx \right)^{1/2}, \tag{3.1}$$

where u is a 2π-periodic function (which may be a constant so no zero-mean condition is assumed), and $a \in L^1(0, 2\pi)$. Here $K(\alpha)$ denotes a sharp constant and we show below that it depends only on the flux

$$\alpha := \frac{1}{2\pi} \int_0^{2\pi} a(x)dx, \tag{3.2}$$

and $K(\alpha) < \infty$ if and only if $\alpha \notin \mathbb{Z}$.

Constant magnetic potential. We first consider the case when $a(x) \equiv \alpha \in (0,1)$. Setting

$$A = \left(i\,\frac{d}{dx} - \alpha \right)^2$$

we consider the positive-definite self-adjoint operator

$$\mathbb{A}(\lambda) := A + \lambda I, \qquad \lambda \geq -\min(\alpha^2, (1-\alpha)^2)$$

and its Green's function $G_\lambda(x,\xi)$:

$$\mathbb{A}(\lambda) G_\lambda(x,\xi) = \delta(x-\xi),$$

which is found in terms of the Fourier series

$$G_\lambda(x,\xi) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \frac{e^{in(x-\xi)}}{(n+\alpha)^2 + \lambda}\,,$$

so that

$$G_\lambda(\xi,\xi) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2 + \lambda} =: G(\lambda)\,, \tag{3.3}$$

The series can be summed explicitly (for instance, by the Poisson summation formula)

$$G(\lambda) = \frac{1}{2\sqrt{\lambda}} \cdot \frac{\sinh(2\pi\sqrt{\lambda})}{\cosh(2\pi\sqrt{\lambda}) - \cos(2\pi\alpha)}\,. \tag{3.4}$$

By Theorem 2.2 in [14] with $\theta = 1/2$ (see also Remark 3.1)

$$K(\alpha) = \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \cdot \sup_{\lambda>0} \lambda^\theta G(\lambda) = 2 \sup_{\lambda>0} \sqrt{\lambda} G(\lambda) = \sup_{\varphi>0} F(\varphi), \tag{3.5}$$

where

$$F(\varphi) := \frac{\sinh\varphi}{\cosh\varphi - \cos(2\pi\alpha)}$$

and $\varphi = 2\pi\sqrt{\lambda}$. Next, the derivative

$$\frac{d}{d\varphi} F(\varphi) = \frac{1 - \cosh\varphi\cos(2\pi\alpha)}{(\cosh\varphi - \cos(2\pi\alpha))^2} > 0$$

if $\cos(2\pi\alpha) \leq 0$, that is, if $\alpha \in [1/4, 3/4]$, so that in this case F is increasing and the supremum is 'attained' at infinity, which gives

$$K(\alpha) = 1 \qquad \text{for} \quad \frac{1}{4} \leq \alpha \leq \frac{3}{4}.$$

Otherwise, for $\alpha \in (0, 1/4) \cup (3/4, 1)$ the function $F(\varphi)$ attains a global maximum at

$$\varphi_*(\alpha) = \operatorname{arccosh}\left(\frac{1}{\cos(2\pi\alpha)}\right),$$

which gives

$$K(\alpha) = F(\varphi_*(\alpha)) = \frac{1}{|\sin(2\pi\alpha)|}.$$

Finally, it is clear from the argument as well as from the result that it is $\alpha \bmod(1)$ that really matters.

Non-constant magnetic potential. Now

$$A = \left(i\frac{d}{dx} - a(x)\right)^2, \tag{3.6}$$

and for the flux α defined in (3.2) let

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{i\left((n+\alpha)x - \int_0^x a(y)dy\right)}.$$

Then $\{\varphi_n\}_{n=-\infty}^{\infty}$ is an orthonormal system in $L_2(0, 2\pi)$. Note that since $n \in \mathbb{Z}$, these functions are periodic and also satisfy the equation

$$\left(i\frac{d}{dx} - a(x)\right)\varphi_n = -(n+\alpha)\varphi_n$$

and therefore we also have

$$A\varphi_n = (n+\alpha)^2\varphi_n.$$

In addition, the system $\{\varphi_n\}_{n=-\infty}^{\infty}$ is complete (since $\varphi_n(x) = c(x)e^{-inx}$ with $|c(x)| = 1/\sqrt{2\pi}$). Then the Green's function for the operator $A + \lambda I$ equals

$$G_\lambda(x,\xi) = \sum_{n\in\mathbb{Z}} \frac{\varphi_n(x-\xi)}{(n+\alpha)^2+\lambda} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{e^{i\left((n+\alpha)(x-\xi) - \int_\xi^x a(y)dy\right)}}{(n+\alpha)^2+\lambda}$$

and the expression for $G_\lambda(\xi,\xi)$ is exactly the same as in (3.3) and therefore everything after (3.3) is the same as in the case of a constant magnetic potential.

Thus, we have proved the following result.

Theorem 3.1. *Inequality* (3.1) *holds for* $\alpha \notin \mathbb{Z}$ *and the sharp constant* $K(\alpha)$ *is given by* (1.8). *Furthermore, for* $\alpha \bmod(1) \in (0, 1/4) \cup (3/4, 1)$ *there exists a unique extremal function*

$$u_\lambda(x) = \sum_{n\in\mathbb{Z}} \frac{\varphi_n(x)}{(n+\alpha)^2+\lambda} = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \frac{e^{i\left((n+\alpha)x - \int_0^x a(y)dy\right)}}{(n+\alpha)^2+\lambda}\,, \tag{3.7}$$

where

$$\lambda = \lambda(\alpha) := \left[\frac{1}{2\pi}\operatorname{arccosh}\left(\frac{1}{\cos(2\pi\alpha)}\right)\right]^2.$$

There are no extremals for $\alpha \bmod(1) \in [1/4, 3/4]$.

Remark 3.1. In our one-dimensional case and operators with explicitly known spectrum and eigenfunctions it makes sense to give a direct proof of (3.5). In fact, using the Fourier series $u(x) = \sum_{k\in\mathbb{Z}} u_k \varphi_k(x)$ and without loss of generality assuming that $u(x)$ attains its maximum at $x = 0$ we have for an arbitrary $\lambda > 0$ the following inequality

$$\begin{aligned} |u(0)|^2 = \frac{1}{2\pi}\left|\sum_{k\in\mathbb{Z}} u_k\right|^2 = \\ = \frac{1}{2\pi}\left|\sum_{k\in\mathbb{Z}} u_k((k+\alpha)^2+\lambda)^{1/2}((k+\alpha)^2+\lambda)^{-1/2}\right|^2 \le \\ \le \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\frac{1}{(k+\alpha)^2+\lambda)}\sum_{k\in\mathbb{Z}}(|u_k|^2((k+\alpha)^2+\lambda)) = \\ = G(\lambda)(\|A^{1/2}u\|^2 + \lambda\|u\|^2), \end{aligned}$$

which turns into equality for $u(x)$ as in (3.7). For $\lambda_* = \|A^{1/2}u\|^2/\|u\|^2$ we see that

$$\|A^{1/2}u\|^2 + \lambda_*\|u\|^2 = 2\lambda_*^{1/2}\ \|A^{1/2}u\|\|u\|$$

and therefore

$$\|u\|_\infty^2 \le 2\sup_{\lambda>0}\lambda^{1/2}G(\lambda)\|A^{1/2}u\|\|u\|,$$

which shows that $K(\alpha) \le 2\sup_{\lambda>0}\lambda^{1/2}G(\lambda)$. To see that we have equality here, we first assume that the supremum is attained at a finite point $\lambda_* < \infty$. Then

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k+\alpha)^2+\lambda_*} = 2\lambda_* \sum_{k\in\mathbb{Z}} \frac{1}{\bigl((k+\alpha)^2+\lambda_*\bigr)^2} \ \left[= 2\lambda_*\|u_{\lambda_*}\|^2\right],$$

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k+\alpha)^2+\lambda_*} = 2 \sum_{k\in\mathbb{Z}} \frac{(k+\alpha)^2}{\bigl((k+\alpha)^2+\lambda_*\bigr)^2} \ \left[= 2\|A^{1/2}u_{\lambda_*}\|^2\right],$$

where the first equality is $(\lambda^{1/2}G(\lambda))'_{\lambda=\lambda_*} = 0$, and the validity of the second follows from the fact that the sum of the two equalities is a valid identity. Since the left-hand side is equal to $\sqrt{2\pi}\|u_{\lambda_*}\|_\infty$ and $\lambda_* = \|A^{1/2}u_{\lambda_*}\|^2/\|u_{\lambda_*}\|^2$, recalling (3.3) we obtain

$$\|u_{\lambda_*}\|^2_\infty = \frac{1}{2\pi}\left(\sum_{k\in\mathbb{Z}} \frac{1}{(k+\alpha)^2+\lambda_*}\right)^2 = 2\lambda_*\|u_{\lambda_*}\|^2 G(\lambda_*) =$$
$$= 2\lambda_*^{1/2}G(\lambda_*)\lambda_*^{1/2}\|u_{\lambda_*}\|^2 = 2\left(\lambda_*^{1/2}G(\lambda_*)\right)\|A^{1/2}u_{\lambda_*}\|\|u_{\lambda_*}\|.$$

This proves that $K(\alpha) = 2\sup_{\lambda>0}\lambda^{1/2}G(\lambda)$ if $\lambda_* < \infty$. Now we look at the case when $\lambda_* = \infty$. Let $2\lim_{\lambda\to\infty}\lambda^{1/2}G(\lambda) = K' \ge K(\alpha)$. Setting $H_N(\lambda) = 2\lambda^{1/2}\sum_{|n|\le N}\frac{1}{(n+\alpha)^2+\lambda}$ we see that there exists a sequence $N(j)\to\infty$ and a sequence $\lambda(j)\to\infty$ such that $H_{N(j)}(\lambda(j))\to K'$. Since $H_N(0) = H_N(\infty) = 0$, it follows that $H_{N(j)}(\lambda)$ attains a maximum at a $\lambda_*(j) < \infty$. The previous argument shows that $H_{N(j)}(\lambda_*(j))$ is the sharp constant in our inequality restricted to Span $\{\varphi_n\}_{n=-N(j)}^{N(j)}$. Therefore

$$K(\alpha) \ge \limsup_{j\to\infty} H_{N(j)}(\lambda_*(j)) \ge \lim_{j\to\infty} H_{N(j)}(\lambda(j)) = K'.$$

As we have seen both cases are possible depending on whether $\alpha \in [1/4, 3/4]$ or $\alpha \in (0, 1/4) \cup (3/4, 1)$.

4. Correction term

In the region $\alpha \in [1/4, 3/4]$ no extremals exist and therefore the might be a correction term in (1.7). By symmetry the cases α and $1-\alpha$ are identical, therefore we can and shall assume that $\alpha \in (0, 1/2]$. We now show that the correction term indeed exists. We consider the maximization problem

$$\mathbb{V}(D) := \sup\{|u(0)|^2 \colon \ \|u\|^2 = 1, \ \|A^{1/2}\|^2 = D\}, \ D \ge \alpha^2. \tag{4.1}$$

Similarly to Theorem 2.1 (see also the general result in Theorem 2.3 in [14]) we have

$$\mathbb{V}(D) = \min_{\lambda \ge -\alpha^2} G(\lambda)(\lambda + D). \tag{4.2}$$

We first consider the cases $\alpha = 1/2$ and $\alpha = 1/4$. We recall the elementary inequality (1.11) and its refinement (1.14) obtained in [14] and show that the case $\alpha = 1/2$ or $1/4$ essentially reduces to the proof of (1.14) in [14]. In fact, for $\alpha = 1/4$ the key function (3.4) becomes

$$G(\lambda) = \frac{1}{2\sqrt{\lambda}} \tanh(2\pi\sqrt{\lambda}), \quad \alpha = \frac{1}{4},$$

and therefore

$$\mathbb{V}(D) = \min_{\lambda \ge -1/16} G(\lambda)(\lambda + D) \le \min_{\lambda \ge 0} \frac{1}{2\sqrt{\lambda}} \tanh(2\pi\sqrt{\lambda})(\lambda + D).$$

Up to a constant factor in the argument of tanh the minimum on the right-hand side was estimated in [14] (see (3.103), (3.104) there), where it was shown that

$$\min_{\lambda \ge 0} \frac{1}{2\sqrt{\lambda}} \tanh \frac{\lambda^{1/2}}{2} (\lambda + D) < \sqrt{D}\left(1 - 2e^{-\sqrt{D}}\right) \tag{4.3}$$

Setting here $\mu = 16\pi^2\lambda$ we obtain for $\mathbb{V}(D)$ in (4.1) with $\alpha = 1/4$

$$\mathbb{V}(D) < \sqrt{D}(1 - 2e^{-4\pi\sqrt{D}}). \tag{4.4}$$

The case $\alpha = 1/2$ is similar. Now in (3.4) we have

$$G(\lambda) = \frac{1}{2\sqrt{\lambda}} \tanh(\pi\sqrt{\lambda}), \quad \alpha = \frac{1}{2},$$

and in a totally similar way we find

$$\mathbb{V}(D) \le \min_{\lambda \ge 0} \frac{1}{2\sqrt{\lambda}} \tanh(\pi\sqrt{\lambda})(\lambda + D) < \sqrt{D}\left(1 - 2e^{-2\pi\sqrt{D}}\right).$$

Thus, we have proved the following inequalities

$$\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|(1 - 2e^{-4\pi\|A^{1/2}u\|/\|u\|}), \quad \alpha = 1/4, 3/4, \tag{4.5}$$

$$\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|(1 - 2e^{-2\pi\|A^{1/2}u\|/\|u\|}), \quad \alpha = 1/2. \tag{4.6}$$

The case $\alpha \in (1/4, 3/4)$ can be treated using the general method of [14]. Our goal is to prove the inequality

$$\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|(1 + 2\cos(2\pi\alpha)e^{-2\pi\|A^{1/2}u\|/\|u\|}), \quad 1/4 < \alpha < 3/4, \tag{4.7}$$

which is equivalent to

$$\mathbb{V}(D) \le \sqrt{D}(1 + 2\cos(2\pi\alpha)e^{-2\pi\sqrt{D}}), \quad D \ge \alpha^2. \tag{4.8}$$

In view of (4.2), to prove this inequality it suffices to find such a substitutiuon $\lambda = \lambda_*(D)$ for which

$$G(\lambda_*(D))(\lambda_*(D)+D) \le \sqrt{D}(1+2\cos(2\pi\alpha)e^{-2\pi\sqrt{D}}), \quad D \ge \alpha^2. \tag{4.9}$$

The exact solution $\lambda = \lambda(D)$ for the minimizer, that is,

$$\lambda(D) = \operatorname{argmin}\{G(\lambda)(\lambda + D)\} \tag{4.10}$$

is the inverse function to the function $D = D(\lambda)$

$$D(\lambda) = -\frac{G(\lambda)}{G'(\lambda)} - \lambda.$$

It is impossible to find $\lambda(D)$ explicitly. However, using (3.4) we can find the asymptotic expansion

$$D(\lambda) = \lambda - 4\pi a\lambda^{3/2}e^{-2\pi\sqrt{\lambda}} + O(e^{-4\pi\sqrt{\lambda}}) \quad \text{as } \lambda \to \infty,$$

where

$$a := 2\cos(2\pi\alpha).$$

Therefore the inverse function $\lambda(D)$ (see (4.10)) has the asymptotic behavior

$$\lambda(D) = D + 4\pi a D^{3/2}e^{-2\pi\sqrt{D}} + O(e^{-4\pi\sqrt{D}}) \quad \text{as } D \to \infty,$$

truncating which we set

$$\lambda_*(D) = D(1 + 4\pi a\sqrt{D}e^{-2\pi\sqrt{D}}).$$

Returning to (4.8) we find that

$$G(\lambda_*(D))(\lambda_*(D) + D) = \sqrt{D}\,\Phi(y),$$

where $y := e^{-2\pi\sqrt{D}}$ and

$$\Phi(y) := \frac{1 - ay\log y}{\sqrt{1 - 2ay\log y}} \cdot \frac{1 - y^{2\sqrt{1-2ay\log y}}}{1 + y^{2\sqrt{1-2ay\log y}} - ay^{\sqrt{1-2ay\log y}}}\,.$$

Therefore inequality (4.9) is equivalent to

$$\Phi(y) < 1 + ay \tag{4.11}$$

for $y \in [0, e^{-2\pi\alpha}]$. The function $\Phi(y)$ has the asymptotic expansion

$$\Phi(y) = 1 + ay + (-a^2(\log y)^2/2 - 2 + a^2)y^2 + O(y^3) \quad \text{as } y \to 0^+,$$

in which the coefficient of the quadratic term is negative for all sufficiently small y. Therefore inequality (4.11) holds for all sufficiently small $y \in [0, y_0]$. The graphs of the function $\Phi(y) - (1 + ay)$ on the whole intervals $y \in [0, e^{-2\pi\alpha}]$ for $\alpha = 1/3$, $3/8$, and $1/2$ are shown in Fig. 1. This 'proves' that inequality (4.11) holds for all $y \in [0, e^{-2\pi\alpha}]$

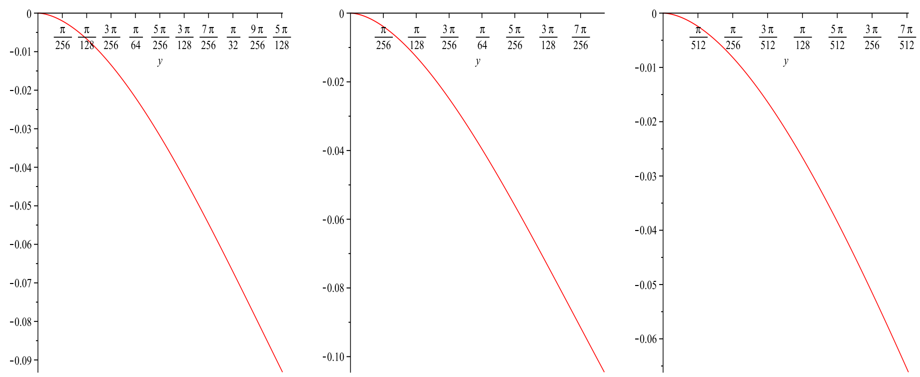

FIGURE 1. Graphs of $\Phi(y) - (1 + ay)$ on $y \in [0, e^{-2\pi\alpha}]$ for $\alpha = 1/3$, $\alpha = 3/8$, and $\alpha = 1/2$; $a = 2\cos(2\pi\alpha)$

and we obtain, as a result, that the following theorem holds.

Theorem 4.1. *For* $\alpha = 1/4$ *and* $\alpha = 3/4$

$$\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|\left(1 - 2e^{-4\pi\|A^{1/2}u\|/\|u\|}\right),$$

while for $\alpha \in (1/4, 3/4)$

$$\|u\|_\infty^2 \le \|A^{1/2}u\|\|u\|\left(1 + 2\cos(2\pi\alpha)e^{-2\pi\|A^{1/2}u\|/\|u\|}\right).$$

All constants are sharp.

5. Carlson–Landau inequalities

One-dimensional inequalities of L_∞-L_2-L_2-type with various boundary conditions are closely connected with Carlson–Landau inequalities and their various improvements.

Carlson–Landau inequality with correction term. In the next theorem we show that both inequality (1.7) in the symmetric case $\alpha = 1/2$, and inequality (1.11) are equivalent to (1.10), while their refined forms (1.14) and (4.6) are, in fact, equivalent and provide a sharp exponential correction term to Landau's improvement of Carlson's inequality. A sharp second-order Carlson-type inequality in the flavor of (1.10) is also given. The notation introduced in (1.16) is used in the following theorem.

Theorem 5.1. *The following inequality holds*

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 \le \pi\|a\|\|a\|_1\left(1-2e^{-2\pi\|a\|_1/\|a\|}\right), \tag{5.1}$$

where all constants are sharp and no extremals exist.

In the second-order case it holds

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 \le \frac{\sqrt{2}\,\pi}{\sqrt[4]{27}}\coth(\pi/2)\|a\|^{3/2}\|a\|_2^{1/2}. \tag{5.2}$$

Inequality (5.2) *saturates at a unique extremal*

$$a_k = \frac{1}{(2k-1)^4+4}\,, \tag{5.3}$$

Proof. Given a non-negative sequence $\{a_k\}_{k=1}^{\infty}$ we construct the sequence $\{b_k\}_{k=-\infty}^{\infty}$ by setting for $k=0,1,\dots,$

$$b_0 = b_{-1} := a_1,\ b_1 = b_{-2} := a_2, \dots, b_k = b_{-(k+1)} := a_{k+1}, \dots .$$

Then for a periodic function

$$u(x) = \sum_{k\in\mathbb{Z}} b_k e^{ikx}$$

we have

$$\|u\|_\infty = \sum_{k\in\mathbb{Z}} b_k = 2\sum_{k=1}^{\infty} a_k, \quad \|u\|^2 = 2\pi\sum_{k\in\mathbb{Z}} b_k^2 = 4\pi\sum_{k=1}^{\infty} a_k^2,$$

and

$$\|A^{1/2}u\|^2 = 2\pi\sum_{k=0}^{\infty}(k+1/2)^2 b_k^2 + 2\pi\sum_{k=-1}^{-\infty}(k+1/2)^2 b_k^2 =$$

$$= 2\pi\sum_{k=1}^{\infty}(k-1/2)^2 a_k^2 + 2\pi\sum_{k=1}^{\infty}(k-1/2)^2 b_{-k}^2 = 4\pi\sum_{k=1}^{\infty}(k-1/2)^2 a_k^2.$$

Substituting this into the second inequality in (4.6) gives (5.1).

An alternative and a simpler proof was given in § 1 by using (1.11) and its refinement (1.14).

As for the second-order inequality, for $u \in H^2(0,L)\cap H^1_0(0,L)$ we have the sharp inequality [14, Theorem 3.9]

$$\|u\|_\infty^2 \le \frac{\sqrt{2}}{\sqrt[4]{27}}\coth(\pi/2)\|u\|^{3/2}\|u''\|^{1/2}, \tag{5.4}$$

saturating at a unique extremal function (for $L = 1$)

$$u_*(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin(2k-1)\pi x}{(2k-1)^4 + 4}\,. \tag{5.5}$$

Setting $L = 1$, substituting $u(x)$ from (1.12) into (5.4) and taking into account that $\|u''\|^2 = 16\pi^4 \sum_{k=1}^{\infty}(k-1/2)^4 a_k^4$ we obtain inequality (5.1), while the unique extremal (5.3) is produced by (5.5).

We finally observe that unlike all the previous Carlson-type inequalities (namely, (1.1), (1.5), (1.6), (1.10), (5.1)) inequality (5.2) has a unique extremal (5.3). □

Intermediate Carlson–Landau inequalities. In conclusion we consider the family of intermediate Carlson–Landau-type inequalities (1.17) in the whole range $\alpha \in [0, 1)$. In the case $\alpha = 1/2$ the Carlson–Landau inequality was supplemented with an exponentially small remainder term in Theorem 5.1.

We now consider the region $\alpha \in [0, 1/2)$. Obviously, $k(\alpha) = \pi$ and we show below that there exists a (sharp) correction term:

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 \le \pi \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty}(k-\alpha)^2 a_k^2\right)^{1/2} - (1-2\alpha)\sum_{k=1}^{\infty} a_k^2. \tag{5.6}$$

For $\alpha = 0$ it is the classical Carlson inequality supplemented with a lower order term in (1.5).

To prove (5.6) we apply our method directly to sequences without going over to functions. We consider the variational problem: for $D \ge (1-\alpha)^2$ find

$$\mathbb{V}(D,\alpha) := \sup\left\{\left(\sum_{k=1}^{\infty} a_k\right)^2 : \ \sum_{k=1}^{\infty} a_k^2 = 1,\ \sum_{k=1}^{\infty}(k-\alpha)^2 a_k^2 = D\right\}. \tag{5.7}$$

In complete analogy with (2.23) and Theorem 2.1 we find that

$$\mathbb{V}(D,\alpha) = \min_{\alpha \ge -(1-\alpha)^2} (\lambda + D) G(\lambda), \tag{5.8}$$

where

$$G(\lambda) = \sum_{k=1}^{\infty} \frac{1}{(k-\alpha)^2 + \lambda}. \tag{5.9}$$

Using the Euler ψ-function $\psi(z)=\frac{d}{dz}\log\Gamma(z)$ and its representation

$$\psi(z)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right)$$

we factorize the denominator in (5.9) and find

$$G(\lambda)=\frac{i(\psi(1-\alpha-i\sqrt{\lambda})-\psi(1-\alpha+i\sqrt{\lambda}))}{2\sqrt{\lambda}}=:\frac{1}{2\sqrt{\lambda}}F(\alpha,\lambda)\,. \tag{5.10}$$

Using the Stirling expansion for the ψ-function

$$\psi(z)=\ln z-\frac{1}{2z}-\frac{1}{12z^2}+O(z^{-3}),$$

we get as $\lambda\to\infty$

$$G(\lambda)=\frac{\pi}{2}\lambda^{-1/2}-\frac{1}{2}(1-2\alpha)\lambda^{-1}+O(\lambda^{-2}).$$

For the unique point of a minimum $\lambda(D)$ in (5.8) we have the equation

$$D=-\frac{G(\lambda)}{G'(\lambda)}-\lambda=\lambda+\frac{2(1-2\alpha)}{\pi}\lambda^{1/2}+\frac{4(1-2\alpha)^2}{\pi^2}+O(\lambda^{-1/2}),$$

giving

$$\lambda(D)=D-\frac{2(1-2\alpha)}{\pi}D^{1/2}-2(1-2\alpha)^2+O(D^{-1/2}). \tag{5.11}$$

Substituting this into $\mathbb{V}(D,\alpha)=-\frac{G(\lambda(D))^2}{G'(\lambda(D))}$ we obtain the expansion

$$\mathbb{V}(D,a)=\pi D^{1/2}-(1-2a)-\frac{(2a-1)^2}{2\pi}D^{-1/2}+O(D^{-1}).$$

The third term here is negative, hence

$$\mathbb{V}(D,a)<\pi D^{1/2}-(1-2a) \tag{5.12}$$

for all sufficiently large D. To see that this inequality holds for all D we truncate the expansion (5.11) by setting

$$\lambda_*(D):=D-\frac{2(1-2\alpha)}{\pi}D^{1/2},$$

and consider the explicitly given function

$$\mathbb{V}_*(D,\alpha):=(\lambda_*(D)+D)G(\lambda_*(D)).$$

Since by definition $\mathbb{V}(D,\alpha)\le\mathbb{V}_*(D,\alpha)$, to establish (5.12) for all D it suffices to show that the following function is negative

$$R(D,\alpha):=\mathbb{V}_*(D,\alpha)-\pi D^{1/2}+(1-2a)$$

for all $D \geq (1-\alpha)^2$. We have the asymptotic expansion

$$R(D,\alpha) = -(1-2\alpha)^2 D^{-1/2} + O(D^{-1})$$

giving that $R(D,\alpha) < 0$ for all sufficiently large D. The graphs of $R(D,\alpha)$ for different α are shown in Fig. 2, where one can see a very rapid convergence to 0 for $\alpha = 1/2$.

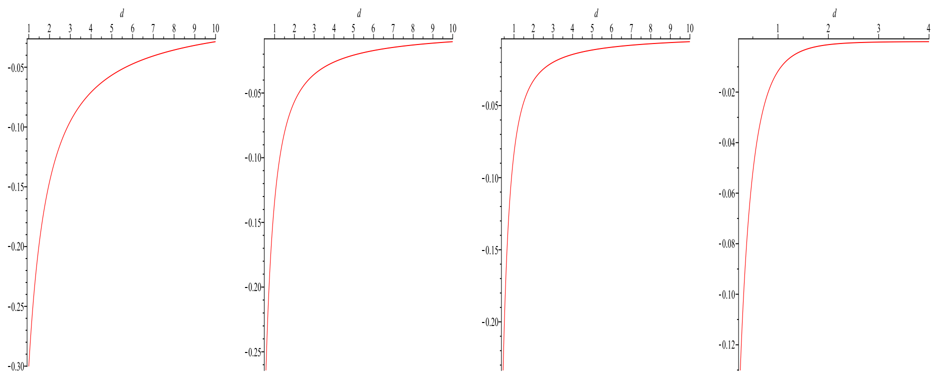

FIGURE 2. Graphs of $R(D,\alpha)$ for $\alpha = 0$, $\alpha = 1/4$, $\alpha = 1/3$, and $\alpha = 1/2$.

The case when $\alpha \in (1/2, 1)$ is qualitatively different and very similar to the 'magnetic' inequalities in Theorem 3.1. Namely, $k(\alpha) > \pi$ and there exists a (unique) extremal in (1.17). In fact, repeating word for word the argument in Remark 3.1 (replacing $\sum_{k\in\mathbb{Z}}$ by $\sum_{k\in\mathbb{N}}$) we obtain that

$$k(\alpha) = 2\sup_{\lambda>0}\sqrt{\lambda}G(\lambda) = \sup_{\lambda>0}F(\alpha,\lambda),$$

where F is defined in (5.10). Since $\lim_{\lambda\to\infty}F(\alpha,\lambda) = \pi$, it follows that $k(\alpha) \geq \pi$. The supremum is, in fact, a maximum, that is attained at a (unique) point $\lambda_*(\alpha)$, for which $\lambda_*(\alpha) \sim (1-\alpha)^2$ and $k(\alpha) \sim 1/(1-\alpha)$ as $\alpha \to 1^-$ (this easily follows from the asymptotic behavior of $\psi(z)$ near 0: $\psi(z) = -\gamma - 1/z + O(z^2)$), see Fig. 3.

Thus, with the help of reliable computer calculations we obtain the following result.

Theorem 5.2. *Inequality* (5.6) *holds for* $\alpha \in [0, 1/2)$. *The constants are sharp, no extremals exist.*

For $\alpha \in (1/2, 1)$ *the sharp constant in* (1.17) *is*

$$k(\alpha) = \max_{\lambda>0} i\big(\psi(1-\alpha-i\sqrt{\lambda}) - \psi(1-\alpha+i\sqrt{\lambda})\big).$$

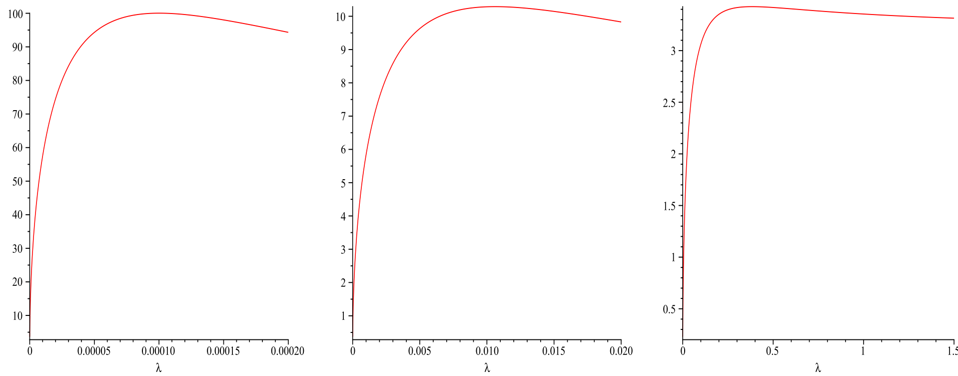

FIGURE 3. Global maximums of $F(\alpha,\lambda)$ for $\alpha=0.99$, $\alpha=0.9$, $\alpha=0.6$.

The maximum is attained at a (unique) point $\lambda_*(\alpha)$ and there exists a unique extremal

$$a_k=\frac{1}{(k-\alpha)^2+\lambda_*(\alpha)}.$$

6. Lieb–Thirring estimates for magnetic Schrödinger operators

One-dimensional Sobolev inequalities for matrices. In this section we give an alternative proof of the main result in [4] along with its generalization to higher order derivatives and magnetic operators.

Let $\{\phi_n\}_{n=1}^N$ be an orthonormal family of vector-functions

$$\phi_n(x)=(\phi_n(x,1),\dots,\phi_n(x,M))^T$$

and

$$(\phi_n,\phi_m)=\sum_{j=1}^M\int_D\phi_n(x,j)\overline{\phi_m(x,j)}dx=\int_D\phi_n(x)^T\overline{\phi_m(x)}dx=\delta_{nm}.$$

Here $D=\mathbb{R}$ or $D=\mathbb{S}^1$. In the latter case we assume that for all n and j

$$\int_0^{2\pi}\phi_n(x,j)dx=0.$$

We consider the $M\times M$ matrix $U(x,y)$

$$U(x,y)=\sum_{n=1}^N\phi_n(x)\overline{\phi_n(y)}^T \tag{6.1}$$

so that $[U(x,y)]_{jk} = \sum_{n=1}^N \phi_n(x,j)\overline{\phi_n(y,k)}$. Clearly,

$$U(x,y)^* = U(y,x)$$

and by orthonormality

$$\int_D U(x,y)U(y,z)dy = \sum_{n,n'=1}^N \int_D \phi_n(x)\overline{\phi_n(y)}^T \phi_{n'}(y)\overline{\phi_{n'}(z)}^T dy =$$

$$= \sum_{n=1}^N \phi_n(x)\overline{\phi_n(z)}^T = U(x,z).$$

In addition, $U(x,x)$ is positive semi-definite, since $(U(x,x)a,a) = \sum_{n=1}^N |a^T\phi_n(x)|^2 \geq 0$.

Theorem 6.1. *Let $m > 1/2$. Then*

$$\int_D \mathrm{Tr}[U(x,x)^{2m+1}]dx \leq C(m)^{2m} \sum_{n=1}^N \sum_{j=1}^M \int_D |\phi_n^{(m)}(x,j)|^2 dx, \tag{6.2}$$

where $C(m)$ is defined in (1.2)*. In particular, for $m = 1, 2$*

$$\int_D \mathrm{Tr}[U(x,x)^3]dx \leq \sum_{n=1}^N \sum_{j=1}^M \int_D |\phi_n'(x,j)|^2 dx,$$

$$\int_D \mathrm{Tr}[U(x,x)^5]dx \leq \frac{4}{27} \sum_{n=1}^N \sum_{j=1}^M \int_D |\phi_n''(x,j)|^2 dx.$$

Proof. We first consider the periodic case. We write

$$\tilde{U}(n,x) = \int_0^{2\pi} \frac{e^{-iyn}}{\sqrt{2\pi}} U(y,x)dy$$

so that

$$U(y,x) = \sum_{k\in\mathbb{Z}_0} \frac{e^{iyk}}{\sqrt{2\pi}} \tilde{U}(k,x) \ .$$

We have

$$\sum_{k\in\mathbb{Z}_0} \tilde{U}(k,x)^*\tilde{U}(k,x) = \int_0^{2\pi} U(y,x)^*U(y,x)dy = U(x,x), \tag{6.3}$$

where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, and we further have

$$\sum_{k\in\mathbb{Z}_0} |k|^{2m}\tilde{U}(k,x)^*\tilde{U}(k,x) = \int_0^{2\pi} [\partial_y^{(m)}U(y,x)]^*\partial_y^{(m)}U(y,x)dy$$

so that by orthonormality

$$\operatorname{Tr}\left[\int_0^{2\pi}\sum_{k\in\mathbb{Z}}|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)dx\right]=$$

$$=\operatorname{Tr}\left[\int_0^{2\pi}\int_0^{2\pi}\sum_{n,n'=1}^N \phi_{n'}^{(m)}(y)\overline{\phi_{n'}(x)}^T\phi_n(x)\overline{\phi_n^{(m)}(y)}^T\,dxdy\right]=$$

$$\operatorname{Tr}\left[\int_0^{2\pi}\sum_{n=1}^N \phi_n^{(m)}(y)\overline{\phi_n^{(m)}(y)}^T\,dy\right]=\sum_{n=1}^N\sum_{j=1}^M\int_0^{2\pi}|\phi_n^{(m)}(x,j)|^2dx. \tag{6.4}$$

Now consider

$$\operatorname{Tr}[U(x,x)^{2m+1}]=\sum_{k\in\mathbb{Z}_0}\operatorname{Tr}[U(x,x)^{2m}\tilde U(k,x)]\frac{e^{ixk}}{\sqrt{2\pi}}=$$

$$\sum_{k\in\mathbb{Z}_0}\operatorname{Tr}\Big[[|k|^{2m}I+\Lambda(x)^{2m}]^{-1/2}U(x,x)^{2m}\tilde U(k,x)\times$$

$$[|k|^{2m}I+\Lambda(x)^{2m}]^{1/2}\Big]\frac{e^{ixk}}{\sqrt{2\pi}}\,,$$

where $\Lambda(x)$ is an arbitrary positive definite matrix. Using below the Cauchy–Schwarz inequality for matrices we get the upper bounds

$$\operatorname{Tr}[U(x,x)^{2m+1}]\le$$

$$\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}_0}\big|\operatorname{Tr}\big[[|k|^{2m}I+\Lambda(x)^{2m}]^{-1/2}U(x,x)^{2m}\ \times$$

$$\tilde U(k,x)[|k|^{2m}I+\Lambda(x)^{2m}]^{1/2}\big]\big|\le$$

$$\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}_0}\big(\operatorname{Tr}\big[U(x,x)^{2m}[|k|^{2m}I+\Lambda(x)^{2m}]^{-1}U(x,x)^{2m}\big]\big)^{1/2}\times$$

$$\Big(\operatorname{Tr}\Big[[|k|^{2m}I+\Lambda(x)^{2m}]\tilde U(k,x)^*\tilde U(k,x)\Big]\Big)^{1/2}\le$$

$$\frac{1}{\sqrt{2\pi}}\left(\sum_{k\in\mathbb{Z}_0}\operatorname{Tr}\big[U(x,x)^{2m}[|k|^{2m}I+\Lambda(x)^{2m}]^{-1}U(x,x)^{2m}\big]\right)^{1/2}\times$$

$$\left(\sum_{k\in\mathbb{Z}_0}\operatorname{Tr}\Big[[|k|^{2m}I+\Lambda(x)^{2m}]\tilde U(k,x)^*\tilde U(k,x)\Big]\right)^{1/2}.$$

For the first factor we have

$$\sum_{k\in\mathbb{Z}_0} \mathrm{Tr}\left[U(x,x)^{2m}[|k|^{2m}I+\Lambda(x)^{2m}]^{-1}U(x,x)^{2m}\right] =$$
$$\mathrm{Tr}\left[U(x,x)^{2m}\sum_{k\in\mathbb{Z}_0}[|k|^{2m}I+\Lambda(x)^{2m}]^{-1}U(x,x)^{2m}\right] <$$
$$2\pi c(m)\,\mathrm{Tr}\left[U(x,x)^{2m}\Lambda(x)^{-(2m-1)}U(x,x)^{2m}\right],$$

where we have used the matrix inequality

$$\sum_{k\in\mathbb{Z}_0}[|k|^{2m}I+\Lambda(x)^{2m}]^{-1} < 2\pi c_0(m)\Lambda(x)^{-(2m-1)}, \quad c_0(m)=\frac{1}{2m\sin\frac{\pi}{2m}}. \tag{6.5}$$

In fact, the action of the matrix on the left-hand side on each eigenvector $e=e(x)$ of $\Lambda(x)$ with eigenvalue $\lambda=\lambda(x)>0$ from the orthonormal basis $\{e_j(x),\ \lambda_j(x)\}_{j=1}^M$ results in multiplication of it by the number $\sum_{k\in\mathbb{Z}_0}\frac{1}{|k|^{2m}+\lambda^{2m}}$ for which we have

$$\sum_{k\in\mathbb{Z}_0}\frac{1}{|k|^{2m}+\lambda^{2m}} = \lambda^{-(2m-1)}\frac{1}{\lambda}\sum_{k\in\mathbb{Z}_0}\frac{1}{(|k|/\lambda)^{2m}+1} <$$
$$\lambda^{-(2m-1)}\,2\int_0^\infty\frac{dx}{x^{2m}+1} = \lambda^{-(2m-1)}2\pi c_0(m),$$

since the function $1/(x^{2m}+1)$ is monotone decreasing on $[0,\infty)$.

For the second factor we simply have

$$\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[[|k|^{2m}I+\Lambda(x)^{2m}]\tilde U(k,x)^*\tilde U(k,x)\right] =$$
$$\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)\right] + \sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[\Lambda(x)^{2m}\tilde U(k,x)^*\tilde U(k,x)\right].$$

If we now chose $\Lambda(x)=\beta(U(x,x)+\varepsilon I)$ and let $\varepsilon\to 0$ we obtain (observing that $\lambda^{4m}/(\lambda+\varepsilon)^{2m-1}\to\lambda^{2m+1}$ as $\varepsilon\to 0$ for $\lambda\ge 0$; this is required in case when $U(x,x)$ is not invertible)

$$\mathrm{Tr}[U(x,x)^{2m+1}] \le c_0(m)^{1/2}\beta^{-(2m-1)/2}\,\mathrm{Tr}[U(x,x)^{2m+1}]^{1/2}\times$$
$$\left(\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)\right] + \beta^{2m}\,\mathrm{Tr}[U(x,x)^{2m+1}]\right),$$

where we have also used (6.3), or

$$\mathrm{Tr}[U(x,x)^{2m+1}] \le$$
$$c_0(m)\biggl(\beta^{-(2m-1)}\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)\right]+\beta\,\mathrm{Tr}[U(x,x)^{2m+1}]\biggr).$$

If we optimize over β, we obtain

$$\mathrm{Tr}[U(x,x)^{2m+1}] \le c_0(m)\frac{1}{\theta^\theta(1-\theta)^{1-\theta}}\times$$
$$\Bigl(\mathrm{Tr}[U(x,x)^{2m+1}]\Bigr)^\theta\biggl(\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)\right]\biggr)^{1-\theta}$$

or

$$\mathrm{Tr}[U(x,x)^{2m+1}] \le C(m)^{2m}\sum_{k\in\mathbb{Z}_0}\mathrm{Tr}\left[|k|^{2m}\tilde U(k,x)^*\tilde U(k,x)\right].$$

If we integrate with respect to x and use (6.4), we obtain (6.2).

In the case of $x\in\mathbb{R}$ the proof is similar. We use the Fourier transform instead of the Fourier series and the matrix equality

$$\int_{-\infty}^{\infty}[|p|^{2m}I+\Lambda(x)^{2m}]^{-1}dp = 2\pi c_0(m)\Lambda(x)^{-(2m-1)}$$

instead of (6.5). □

The one-dimensional periodic magnetic case is treated similarly. Suppose that as before we have a family of orthonormal periodic vector-functions (no zero average condition is assumed). As before we construct the matrix U (6.1).

Theorem 6.2. *The following inequality holds*

$$\int_0^{2\pi}\mathrm{Tr}[U(x,x)^3]dx \le K(\alpha)^2\sum_{n=1}^N\sum_{j=1}^M\int_0^{2\pi}|(i\partial_x-a(x))\phi_n(x,j)|^2dx, \tag{6.6}$$

where $K(\alpha)$ is defined in (1.8).

Proof. We define the matrix Fourier coefficients for all $n\in\mathbb{Z}$. We now have

$$\sum_{k\in\mathbb{Z}}\tilde U(k,x)^*\tilde U(k,x) = \int_0^{2\pi}U(y,x)^*U(y,x)dy = U(x,x)\ ,$$

and

$$\sum_{k\in\mathbb{Z}}(k+\alpha)^2\tilde{U}(k,x)^*\tilde{U}(k,x) = \int_0^{2\pi}[(i\partial_y-a)U(y,x)]^*(i\partial_y-a)U(y,x)dy$$

so that instead of (6.4) we now have

$$\int_0^{2\pi}\sum_{k\in\mathbb{Z}}(k+\alpha)^2\tilde{U}(k,x)^*\tilde{U}(k,x)dx = \sum_{n=1}^N\sum_{j=1}^M\int_0^{2\pi}|(i\partial_x-a)\phi_n(x,j)|^2dx. \tag{6.7}$$

As in the proof of Theorem 6.1 we have

$$\mathrm{Tr}[U(x,x)^3] \le \frac{1}{\sqrt{2\pi}}\left(\sum_{k\in\mathbb{Z}}\mathrm{Tr}\left[U(x,x)^2[(k+\alpha)^2I+\Lambda(x)^2]^{-1}U(x,x)^2\right]\right)^{1/2}\times \left(\sum_{k\in\mathbb{Z}}\mathrm{Tr}\left[[(k+\alpha)^2I+\Lambda(x)^2]\tilde{U}(k,x)^*\tilde{U}(k,x)\right]\right)^{1/2}.$$

Now as a matrix inequality

$$\sum_{k\in\mathbb{Z}}[(k+\alpha)^2I+\Lambda(x)^2]^{-1} < \pi K(\alpha)\Lambda(x)^{-1},$$

since the action of the matrix on the left-hand side on an eigenvector $e=e(x)$ of $\Lambda(x)$ with eigenvalue $\lambda=\lambda(x)$ is a multiplication of it by the number $\sum_{k\in\mathbb{Z}}\frac{1}{(k+\alpha)^2+\lambda^2}$ and in view of (3.5)

$$\sum_{k\in\mathbb{Z}}\frac{1}{(k+\alpha)^2+\lambda^2} < \pi K(\alpha)\frac{1}{\lambda}\,.$$

If we again set $\Lambda(x)=\beta(U(x,x)+\varepsilon I)$ and let $\varepsilon\to 0$ we obtain

$$\mathrm{Tr}[U(x,x)^3] \le \le \frac{K(\alpha)}{2}\left(\beta^{-1}\sum_{k\in\mathbb{Z}}(k+\alpha)^2\,\mathrm{Tr}[\tilde{U}(k,x)^*\tilde{U}(k,x)]+\beta\,\mathrm{Tr}[U(x,x)^3]\right).$$

If we optimize over β, we get

$$\mathrm{Tr}[U(x,x)^3] \le \le K(\alpha)\left(\sum_{k\in\mathbb{Z}}(k+\alpha)^2\,\mathrm{Tr}[\tilde{U}(k,x)^*\tilde{U}(k,x)]\right)^{1/2}\left(Tr[U(x,x)^3]\right)^{1/2}$$

and hence

$$\mathrm{Tr}[U(x,x)^3] \le K(\alpha)^2 \sum_{k\in\mathbb{Z}} (k+\alpha)^2\,\mathrm{Tr}[\tilde U(k,x)^*\tilde U(k,x)]$$

from which our inequality follows by integration in x and using (6.7). $\square$

Remark 6.1. In the scalar case $M=1$ inequality (6.6) becomes

$$\int_0^{2\pi}\left(\sum_{n=1}^N |\phi_n(x)|^2\right)^3 dx \le K(\alpha)^2 \sum_{n=1}^N \int_0^{2\pi} |(i\partial_x - a(x))\,\phi_n(x)|^2\,dx \tag{6.8}$$

and follows from (3.1) by the method of [5].

Theorem 6.2 is equivalent to the estimate of the negative trace of the magnetic Schrödinger operator

$$H = \left(i\frac{d}{dx} - a(x)\right)^2 - V \tag{6.9}$$

in $L_2(\mathbb{S}^1)$ with matrix-valued potential V.

Theorem 6.3. *Let $V \ge 0$ be a $M\times M$ Hermitian matrix such that $\mathrm{Tr}\,V^{3/2} \in L_1(\mathbb{R})$. Then the spectrum of operator* (6.9) *is discrete and the negative eigenvalues $-\lambda_n \le 0$ satisfy the estimate*

$$\sum_n \lambda_n \le \frac{2}{3\sqrt{3}} K(\alpha) \int_0^{2\pi} \mathrm{Tr}[V(x)^{3/2}]dx. \tag{6.10}$$

Proof. (See [4].) Let $\{\phi_n\}_{n=1}^N$ be the orthonormal eigen-vector functions corresponding to $\{-\lambda_n\}_{n=1}^N$:

$$\left(i\frac{d}{dx} - a(x)\right)^2 \phi_n - V\phi_n = -\lambda_n\phi_n.$$

Then, using (6.6) and Hölder's inequality for traces

$$\mathrm{Tr}[AB] \le (\mathrm{Tr}([(A^*A)^{p/2}])^{1/p} (\mathrm{Tr}([(B^*B)^{p'/2}])^{1/p'}$$

and setting below $X := \int_0^{2\pi} \operatorname{Tr}[U(x,x)^3]dx$ we obtain

$$\sum_{n=1}^N \lambda_n =$$
$$-\sum_{n=1}^N \sum_{j=1}^M \int_0^{2\pi} |(i\partial_x - a(x))\phi_n(x,j)|^2 dx + \int_0^{2\pi} \operatorname{Tr}[V(x)U(x,x)]dx \leq$$
$$\leq \left(\int_0^{2\pi} \operatorname{Tr}[V(x)^{3/2}]dx \right)^{2/3} X^{1/3} - K(\alpha)^{-2} X.$$

Calculating the maximum with respect to X we obtain (6.10). □

Let

$$L^{\mathrm{cl}}_{\gamma,d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1-|\xi|^2)^\gamma_+ \, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + d/2 + 1)}. \tag{6.11}$$

By using the Aizenmann-Lieb argument [1] we immediately obtain the following statement for the Riesz means of the eigenvalues for magnetic Schrödinger operators with matrix-valued potentials.

Corollary 6.1. *Let $V \geq 0$ be a smooth $M \times M$ Hermitian matrix, such that $\operatorname{Tr} V^{\gamma+1/2} \in L_1(0,2\pi)$. Then for any $\gamma \geq 1$ the negative eigenvalues of the operator* (6.9) *satisfy the inequalities*

$$\sum \lambda_n^\gamma \leq L_{\gamma,1} \int_0^{2\pi} \operatorname{Tr}[V(x)^{1/2+\gamma}] \, dx,$$

where

$$L_{\gamma,1} \leq \frac{2}{3\sqrt{3}} K(\alpha) \frac{L^{\mathrm{cl}}_{\gamma,1}}{L^{\mathrm{cl}}_{1,1}} = \frac{\pi}{\sqrt{3}} K(\alpha) L^{\mathrm{cl}}_{\gamma,1}.$$

Proof. It is enough to prove this result for smooth matrix-valued potentials. Note that Theorem 6.3 is equivalent to

$$\sum_n \lambda_n \leq \frac{2}{3\sqrt{3}} K(\alpha) (L^{\mathrm{cl}}_{1,1})^{-1} \int_0^{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}\left[\left(|\xi|^2 - V(x) \right)_- \right] \frac{d\xi dx}{2\pi}.$$

Scaling gives the simple identity for all $s \in \mathbb{R}$

$$s_-^\gamma = C_\gamma \int_0^\infty t^{\gamma-2} (s+t)_- dt, \qquad C_\gamma^{-1} = \mathcal{B}(\gamma-1, 2),$$

where $\mathcal{B}$ is the Beta function. Let $\{\mu_j(x)\}_{j=1}^M$ be eigenvalues of the matrix-function $V(x)$. Then

$$
\begin{aligned}
&\sum_n \lambda_n^\gamma = C_\gamma \sum_n \int_0^\infty t^{\gamma-2}(-\lambda_n+t)_- dt \\
&\le \frac{2K(\alpha)}{3\sqrt{3}}\,\frac{C_\gamma}{L_{1,1}^{\mathrm{cl}}} \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty t^{\gamma-2}\,\mathrm{Tr}\left[\left(|\xi|^2 - V(x)+t\right)_-\right]\,\frac{d\xi dx}{2\pi}\,dt \\
&= \frac{2K(\alpha)}{3\sqrt{3}}\,\frac{C_\gamma}{L_{1,1}^{\mathrm{cl}}} \sum_{j=1}^M \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty t^{\gamma-2}\,\mathrm{Tr}\left[\left(|\xi|^2 - \mu_j+t\right)_-\right]\,\frac{d\xi dx}{2\pi}\,dt \\
&= \frac{2K(\alpha)}{3\sqrt{3}}(L_{1,1}^{\mathrm{cl}})^{-1} \int_0^{2\pi} \int_{-\infty}^\infty \mathrm{Tr}\left[\left(|\xi|^2 - V(x)\right)_-^\gamma\right]\,\frac{d\xi dx}{2\pi} \\
&= \frac{2}{3\sqrt{3}}K(\alpha)\frac{L_{\gamma,1}^{\mathrm{cl}}}{L_{1,1}^{\mathrm{cl}}} \int_0^{2\pi} \mathrm{Tr}[V(x)^{1/2+\gamma}]\,dx.
\end{aligned}
$$

□

Magnetic Schrödinger operator in $\mathbb{T}^n \times \mathbb{R}^m$. Let us consider the eigenvalue problem for the Schrödinger operator $\mathcal{H}$ in $L_2(\mathbb{T}^{d_1} \times \mathbb{R}^{d_2})$:

$$
\mathcal{H}\Psi = -\Delta_y \Psi + (i\,\nabla_x - A(x))^2\,\Psi - V(x,y)\Psi = -\lambda\Psi, \qquad (x,y) \in \mathbb{T}^{d_1}\times\mathbb{R}^{d_2}, \quad (6.12)
$$

where $\mathbb{T}^{d_1} = \underbrace{\mathbb{S}^1 \times \mathbb{S}^1}_{d_1-times}$ is the standard torus of dimension d_1 and

$$
A(x) = (a_1(x_1), \dots a_{d_1}(x_{d_1}))
$$

is the magnetic vector potential in the "diagonal" case $a_j(x) = a_j(x_j)$. Assume that

$$
\alpha_j = \frac{1}{2\pi}\int_0^{2\pi} a_j(x_j)\,dx_j \notin \mathbb{Z}, \qquad 1 \le j \le d_1.
$$

Then we have

Theorem 6.4. *Suppose that the potential $V(x,y) \ge 0$ in (6.12) and $V \in L_{\gamma+(d_1+d_2)/2}(\mathbb{T}^{d_1}\times\mathbb{R}^{d_2})$. If $\gamma \ge 1/2$, then the following bound holds for the negative eigenvalues:*

$$
\sum_n \lambda_n^\gamma \le L_{\gamma,d_1+d_2} \int_{\mathbb{T}^{d_1}\times\mathbb{R}^{d_2}} V^{\gamma+(d_1+d_2)/2}(x,y)\,dxdy. \qquad (6.13)
$$

Here, if $d_1, d_2 \geq 1$ and $1/2 \leq \gamma < 1$, then

$$L_{\gamma,d_1+d_2} \leq R_{\gamma,d_1+d_2} := 2\left(\frac{2}{3\sqrt{3}}\right)^{d_1} \frac{L^{\mathrm{cl}}_{\gamma,d_1+d_2}}{(L^{\mathrm{cl}}_{1,1})^{d_1}} \prod_{j=1}^{d_1} K(\alpha_j) =$$

$$= 2\left(\frac{\pi}{\sqrt{3}}\right)^{d_1} L^{\mathrm{cl}}_{\gamma,d_1+d_2} \prod_{j=1}^{d_1} K(\alpha_j),$$

and, if $d_1 \geq 1$, $d_2 \geq 0$ and $\gamma \geq 1$, then

$$L_{\gamma,d_1+d_2} \leq \frac{1}{2} R_{\gamma,d_1+d_2}.$$

Proof. As in Corollary 6.1 it is enough to prove this result for smooth compactly supported potentials. We shall use the so-called "lifting argument with respect to dimensions", see [11].

Let $x = (x_1, x')$ and $y = (y_1, y')$, where $x' \in \mathbb{R}^{d_1-1}$ and $y' \in \mathbb{R}^{d_2-1}$ and let $A(x) = (a_1(x_1), A'(x'))$. Denote

$$-\Delta' = -(\nabla_{y'})^2, \quad -\Delta_A = \left(i\,\nabla_x - A(x)\right)^2, \quad -\Delta_{A'} = \left(i\nabla'_{x'} - A'(x')\right)^2.$$

By applying the result in [8] on the 1/2-moments we have

$$\sum_n \lambda_n^{1/2}(\mathcal{H}) = \sum_n \lambda_n^{1/2}(-\partial^2_{y_1} - \Delta_A - \Delta' - V)$$

$$\leq \sum_n \lambda_n^{1/2}\left(-\partial^2_{y_1} - (-\Delta_A - \Delta' - V)_-\right)$$

$$\leq 2L^{\mathrm{cl}}_{1/2,1} \int_{\mathbb{R}} \mathrm{Tr}\,[-\Delta_A - \Delta' - V]_-\, dy_1$$

$$= 2L^{\mathrm{cl}}_{1/2,1} \sum_n \int_{\mathbb{R}} \lambda_n\left(\left(i\partial_{x_1} - a_1(x_1)\right)^2 - \Delta_{A'} - \Delta' - V(x, y_1, y')\right) dy_1$$

$$\leq 2L^{\mathrm{cl}}_{1/2,1} \sum_n \int_{\mathbb{R}} \lambda_n\Big(\left(i\partial_{x_1} - a_1(x_1)\right)^2 - (-\Delta_{A'} - \Delta' - V(x, y_1, y'))_-\Big) dy_1.$$

Then Theorem 6.3 implies

$$\sum_n \lambda_n^{1/2}(\mathcal{H}) \leq 2L^{\mathrm{cl}}_{1/2,1} L^{\mathrm{cl}}_{1,1} \frac{2}{3\sqrt{3}\, L^{\mathrm{cl}}_{1,1}}\, K(\alpha_1)$$

$$\times \int_{\mathbb{R}} \int_0^{2\pi} \mathrm{Tr}\,[-\Delta_{A'} - \Delta' - V(x, y_1, y')]_-^{3/2}\, dx_1 dy_1.$$

Now we first repeat this argument $d_1 - 1$ times "splitting" the operator $(i\nabla' - A'(x))^2$ and using Corollary 6.1. Then repeat it again $d_2 - 1$

times "splitting" the operator Δ' and using the semiclassical estimates for the γ-Riesz means with $\gamma \geq 3/2$ for the negative eigenvalues of the Schrödinger operators with matrix-valued potentials [11]. Finally we obtain

$$\sum_n \lambda_n^{1/2}(\mathcal{H}) \leq 2 \left(\prod_{l=0}^{d_1+d_2-1} L^{\mathrm{cl}}_{\gamma+l/2,1} \right) \left(\frac{2}{3\sqrt{3}\ L^{\mathrm{cl}}_{1,1}} \right)^{d_1} \prod_{j=1}^{d_1} K(\alpha_j) \\ \times \int_{\mathbb{T}^{d_1}\times\mathbb{R}^{d_2}} V^{\gamma+(d_1+d_2)/2}(x,y)\, dxdy.$$

In order to prove (6.13) for the case $d_1, d_2 \geq 1$ and $1/2 \leq \gamma < 1$, it remains to notice that (see (6.11))

$$\prod_{l=0}^{d_1+d_2-1} L^{\mathrm{cl}}_{\gamma+l/2,1} = L^{\mathrm{cl}}_{\gamma,d_1+d_2}.$$

For the proof of the case $d_1 \geq 1$, $d_2 \geq 0$ and $\gamma \geq 1$ we argue similarly, but we omit the first step in the previous argument starting directly with 1-moments. □

For the special cases $d_1 = d_2 = 1$ and $d_1 = 2$, $d_2 = 0$ we state the following corollary of Theorem 6.4:

Corollary 6.2. *Suppose that the potential $V(x,y) \geq 0$ in* (6.12). *Then for $d_1 = d_2 = 1$ the following bounds hold for the* 1/2- *and* 1-*moments of the negative eigenvalues:*

$$\sum_k \lambda_k^{1/2} \leq \frac{1}{3\sqrt{3}} K(\alpha) \int_{\mathbb{R}\times\mathbb{S}^1} V^{3/2}(x,y)dydx,$$
$$\sum_k \lambda_k \leq \frac{1}{8\sqrt{3}} K(\alpha) \int_{\mathbb{R}\times\mathbb{S}^1} V^{2}(x,y)dydx.$$

For $d_1 = 2$, $d_2 = 0$ we have

$$\sum_k \lambda_k \leq \frac{\pi}{24} K(\alpha_1)K(\alpha_2) \int_{\mathbb{T}^2} V^2(x_1,x_2)dx_1dx_2.$$

Acknowledgements. A.I. and S.Z. acknowledge financial support from the Russian Science Foundation (grant no. 14-21-00025) and M.L.'s research is supported in part by US NSF grant DMS - 1301555.

References

[1] M. Aizenman and E.H. Lieb. On semi-classical bounds for eigenvalues of Schrödinger operators. *Phys. Lett.* **66A** (1978), 427–429.

[2] M.V. Bartuccelli, J. Deane, and S.V. Zelik. Asymptotic expansions and extremals for the critical Sobolev and Gagliardo-Nirenberg inequalities on a torus. *Proc. Roy. Soc. Edinburgh* **143A** (2013), 445–482.

[3] F. Carlson. Une inégalité. *Ark. Mat. Astr. Fysik* **25B** (1934), No. 1, 1–5.

[4] J. Dolbeault, A. Laptev, and M. Loss. Lieb–Thirring inequalities with improved constants. *J. European Math. Soc.* **10**:4 (2008), 1121–1126.

[5] A. Eden and C. Foias. A simple proof of the generalized Lieb–Thirring inequalities in one space dimension. *J. Math. Anal. Appl.* **162** (1991), 250–254.

[6] G.H. Hardy. A note on two inequalities. *J. London Math. Soc.* **11** (1936), 167–170.

[7] G.H. Hardy, J.E. Littlewood, and G.Pólya. *Inequalities,* Cambridge Univ. Press, Cambridge 1934; Addendum by V.I. Levin and S.B. Stechkin to the Russian translation, GIIL, Moscow 1948; English transl. in V.I. Levin and S.B. Stechkin, Inequalities, *Amer. Math. Soc. Transl.*, 14 (1960), 1–22.

[8] D. Hundertmark, A. Laptev, and T. Weidl. New bounds on the Lieb–Thirring constants. *Inventiones Mathematicae*, **140**:3 (2000), 693–704.

[9] A.A. Ilyin. Best constants in multiplicative inequalities for sup-norms. *J. London Math. Soc.*(2) **58**, 84–96 (1998).

[10] A. Laptev and T. Weidl. Hardy inequalities for magnetic Dirichlet forms. Mathematical results in quantum mechanics (Prague, 1998), 299-305, *Oper. Theory Adv. Appl.*, 108, Birkhauser, Basel, 1999.

[11] A. Laptev and T. Weidl. Sharp Lieb–Thirring inequalities in high dimensions. *Acta Math.* **184** (2000), 87–111.

[12] L. Larsson, L. Maligranda, J. Pečarić, and L.–E. Persson. *Multiplicative inequalities of Carlson type and intepolation.* World Scientific, Singapore, 2006.

[13] L.V. Taikov. Kolmogorov-type inequalities and the best formulas for numerical differentiation. *Mat. Zametki* **4**, 233–238 (1968); English transl. *Math. Notes* **4** (1968), 631–634.

[14] S.V. Zelik and A.A.Ilyin. Green's function asymptotics and sharp interpolation inequalities. *Uspekhi Mat. Nauk* **69**:2 (2014), 23–76; English transl. in *Russian Math. Surveys* 69:2 (2014).

Keldysh Institute of Applied Mathematics and Institute for Information Transmission Problems;
Imperial College London and Institute Mittag–Leffler;
School of Mathematics, Georgia Institute of Technology;
University of Surrey, Department of Mathematics and Keldysh Institute of Applied Mathematics

E-mail address: ilyin@keldysh.ru; a.laptev@imperial.ac.uk; loss@math.gatech.edu; s.zelik@surrey.ac.uk