THE NEGATIVE SPECTRUM OF A CLASS OF TWO-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SPHERICALLY SYMMETRIC POTENTIALS

BY

A. LAPTEV

ABSTRACT. We obtain an inequality for the number of the negative eigenvalues of a two-dimensional Schrödinger operator with circular symmetric potentials.

1. Introduction and the main result. Let $H = H_{b,V}$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$, $d \geqslant 2$,

$$H_{b,V} = -\Delta + b|x|^{-2} - V, \qquad x \in \mathbb{R}^d, \quad b \in \mathbb{R}, \tag{1.1}$$

where $V \geqslant 0$ is a locally integrable function in $\mathbb{R}^d$. Denote by $N_b(V)$ the number of the negative eigenvalues of the operator (1.1). If $d \geqslant 3$ and $b > -(d-2)^2/4$, then

$$N_b(V) \leqslant C(b,d) \int_{\mathbb{R}^d} V^{d/2}\, dx \tag{1.2}$$

is known as the Cwikel-Lieb-Rosenblum inequality (see [C],[L] and [R]).

It is also known that if $d = 2$ and $b = 0$, then an arbitrary small perturbation by a nonnegative potential $V \in L^1(\mathbb{R}^2)$ generates at least one negative eigenvalue and therefore one cannot expect the inequality (1.2) to be true.

It was shown in [S1,2] and in a sharper form in [BL] that under some additional conditions on V if $d = 2$ and $b = 0$, then the problem can be separated in two problems. The first one is defined by the restriction of the operater (1.1) to the subspace of functions depending on $|x|$ and thus is reduced to a well studied one-dimensional Schrödinger operator with the potential $\widetilde{V}(r) = \frac{1}{2\pi}\int_{|x|=r} V\, d\theta$. In particular, for this class of operators there are necessary and sufficient conditions in [BS] describing when the asymptotics $N_{b=0}(\alpha\widetilde{V}) = O(\alpha)$, as $\alpha \to \infty$ is true. The second problem is defined by a class of functions whose mean values over $\mathbb{S}^1$ are equal zero. For the function from this subspace we have Hardy's inequality (see [Mz], [OK]) which automatically gives the "supporting" term $b|x|^{-2}$ with some $b > 0$.

All this suggests that in order to study the case $d = 2$, we have to pay special attention to the operator (1.1), where $b > 0$.

1991 *Mathematics Subject Classification.* 35P15, 35P20.

Supported by the Swedish Natural Sciences Research Council, Grant M-AA/MA 09364-320.

Notice, however, that if we "take off" the resonance state at 0 by choosing $b > 0$, then the condition $V \in L^1(\mathbb{R}^2)$ still does not guarantee even the semiboundness of (1.1) from below.

Let $\mathcal{H}^1(\mathbb{R}^2)$ be a so-called homogeneous H^1 class of functions defined by

$$\mathcal{H}^1(\mathbb{R}^2) = \Big\{u : \int \Big(|\nabla u|^2 + \frac{|u|^2}{|x|^2}\Big)\, dx < \infty\Big\}. \tag{1.3}$$

It turns out that for a circular symmetric potential $V = V(|x|)$, $V \in L^1(\mathbb{R}^2)$, the quadratic form

$$h_{b,V}[u] = \int_{\mathbb{R}^2} (|\nabla u|^2 + b|x|^{-2}|u|^2)\, dx - \int_{\mathbb{R}^2} V|u|^2\, dx \tag{1.4}$$

defined on $\mathcal{H}^1(\mathbb{R}^2)$ is semibounded and closed in $L^2(\mathbb{R}^2)$ and, hence, defines a selfadjoint operator $H_{b,V}$.

The aim of this note is to show that for the operator $H_{b,V}$ defined by the quadratic form (1.4), the estimate (1.2) is fulfilled.

Our main result is the following theorem:

Theorem. *Let $d = 2$, $b > 0$ and $V(x) = V(|x|) \geqslant 0$. Then*

$$N_b(V) \leqslant \frac{A(b)}{4\pi} \int_{\mathbb{R}^2} V(x)\, dx,$$

where

$$A(b) = \sup_{\mu>0} \Big\{\mu^{-1/2} \cdot \Big(\#\{n : \, n^2 + b - \mu < 0, \; n \in \mathbb{Z}\}\Big)\Big\}. \tag{1.5}$$

Remark 1. Notice that $A(b) \to \infty$, as $b \to 0$. In particular, $A(b) = \mu^{-1/2}$, if $b < \mu \leqslant 1 + b$.

Remark 2. Theorem gives a simple class of potentials in two-dimensional case where the inequality (1.2) is fulfiled. For its generalization see [LN].

2. An auxiliary result. When proving Theorem we use the limiting case of the Lieb-Thirring inequality for a one-dimensional Schrödinger operator. Namely, let

$$L\, v(t) = -v''(t) - W(t)\, v(t), \qquad W \geqslant 0, \quad t \in \mathbb{R}^1, \tag{2.1}$$

be a selfadjoint operator in $L^2(\mathbb{R})$ whose negative spectrum is descrete. Denote by $\{-\mu_k\}_{k=1}^{\infty}$, the negative eigenvalues of the operator L.

Lemma. *If $W \in L^1(\mathbb{R}^1)$ and $W \geqslant 0$, then*

$$\sum_k \mu_k^{1/2} \leqslant \frac{1}{2} \int W(t)\, dt. \tag{2.2}$$

The constant $1/2$ appearing in the right hand side in (2.2) is sharp and this was recently proved in [HLT]. The upper estimate for $\sum_k \mu_k^{1/2}$ via $\|W\|_{L^1(\mathbb{R}^1)}$ with some constant greater than $1/2$ was first proved in [W]. Notice also that the equality in (2.2) is achieved when $W(t) = \delta(t)$ and we have only one negative eigenvalue equal $(-1/4)$. Both proves obtained in [HLT] and [W] are based on the Birman-Schwinger principle [B], [Sch].

3. Proof of Theorem. Let us consider the quadratic form (1.4) and introduce the polar coordinates $x = (r, \theta)$, $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi]$. Then the form (1.4) can be rewritten as

$$h_{b,V}[u] = \int_0^\infty \int_0^{2\pi} \Big(|u_r'|^2 + r^{-2}\big(|u_\theta'|^2 + (b - r^2 V(r))|u|^2\big)\Big)\, r\, dr\, d\theta. \tag{3.1}$$

Let $\{-\lambda_n\}$ be the negative eigenvalues of H. Then in view of the variational principal we obtain

$$N_b(V) = \#\{n:\, -\lambda_n(H) < 0\} = \dim\{u:\, (Hu, u) \leqslant 0,\, u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})\}.$$

This allows us to assume that $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ when estimating the number of the negative eigenvalues generated by the form (3.1). Changing variables $r = e^t$ and denoting $w(t, \theta) = u(e^t, \theta)$, $t \in \mathbb{R}$, $\theta \in [0, 2\pi]$, we transfer the form (3.1) to

$$\widetilde{h}[w] := \int_{-\infty}^\infty \int_{\mathbb{S}^1} \Big(|w_t'|^2 + \big(|w_\theta'|^2 + (b - \widetilde{V})|w|^2\big)\Big)\, dt\, d\theta, \tag{3.2}$$

where

$$\widetilde{V}(t) = e^{2t} V(e^t). \tag{3.3}$$

Let $\{-\mu_k(\widetilde{V})\}$ and $\{v_k(t)\}$, $k \in \mathbb{N}$ be the eigenvalues and eigenfunctions of the operator (2.1) where $W := \widetilde{V}$. Separating variables we find that the eigenfunctions of the operator defined by the quadratic form $\widetilde{h}$ are equal to $v_k(t)e^{in\theta}$, $n \in \mathbb{Z}$, $k = \mathbb{N}$, and the corresponding eigenvalues are $b + n^2 - \mu_k$. Thus we obtain

$$\begin{aligned} N(V) &= \#\{(k, n):\, b + n^2 - \mu_k,\ k \in \mathbb{N},\, n \in \mathbb{Z}\} \\ &\leqslant A(b) \sum \mu_k^{1/2} \leqslant \frac{A(b)}{2} \int_{-\infty}^\infty \widetilde{V}(t)\, dt = \frac{A(b)}{2} \int_{-\infty}^\infty V(e^t)\, e^{2t}\, dt \\ &= \frac{A(b)}{4\pi} \int_0^\infty \int_0^{2\pi} V(r)\, r\, dx = \frac{A(b)}{4\pi} \int_{\mathbb{R}^2} V(x)\, dx, \end{aligned}$$

where $A(b)$ is defined in (1.5).

The proof is complete. □

Acknowledgement. The author is grateful to M.Solomyak for the invitation to the Weizmann Institute (Rehovot, Israel) where the author was able to complete this paper.

References

[B] M.Sh.Birman, *On the spectrum of singular boundary-value problems.*, Mat. Sb. **55** (1961), 125–174 (Russian); English transl. in Amer. Math. Soc. Trans. **53** (1966), 23–80.

[BL] M.Sh.Birman and A.Laptev, *The negative discrete spectrum of a two-dimensional Schrödinger operator*, Comm. Pure Appl. Math. **XLIX** (1996), 967–997.

[BS] M.Sh.Birman and M.Z.Solomyak, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations*, Advances in Soviet Math. **7** (1991), 1–55.

[C] M.Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators.*, Ann. Math. **106** (1977), 93–100.

[HLT] D.Hundertmark, E.H.Lieb and L.E.Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator.*, Adv. Theor. Math. Phys. **2** (1998), 719–731.

[LN] A.Laptev and Yu.Netrusov, *On the negative eigenvalues of a class of Schrödinger operators*, AMS Translations, Series 2, **189** (1998), 178–186.

[L] E.H.Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators.*, Bull. of the AMS. **82** (1976), 751–753.

[Mz] V. Maz'ya, *Sobolev Spases*, Springer-Verlag, Berlin-Heidelberg-New York-Tokio, 1985.

[OK] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, 1990.

[R] G.V.Rosenblum, *Distribution of the discrete spectrum of singular differential operators.*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (1976), 75-86 (Russian); English transl. in Soviet Math. Izv. VUZ **20** (1976), 63–71.

[Sch] J.Schwinger, *On the bound states of a given potential.*, Proc.Nat.Acad.Sci. **47** (1961), 122-129.

[S1] M.Solomyak, *Piecewise-polynomial approximation of functions from* $H^l((0,1)^d)$, $2l = d$, *and applications to the spectral theory of Schrödinger operator*, Israel J. Math. **86** (1994), 253–276.

[S2] M.Solomyak, *Spectral problems related to the critical exponent in the Sobolev embedding theorem*, Proc. London Math. Soc. **71** (1995), 53–75.

[W] T.Weidl, *On the Lieb-Thirring constants* $L_{\gamma,1}$ *for* $\gamma \geqslant 1/2$., Comm. Math. Phys. **178** (1996), 135–146.

Department of Mathematics,
Royal Institute of Technology,
S-100 44 Stockholm, Sweden
E-mail address: laptev@math.kth.se