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On Inequalities for the Bound States of Schrödinger Operators

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Abstract

We improve the Lieb constant in the Cwikel-Lieb-Rozenblum inequality for the number of bound states of Schrödinger operators whose potential equals the characteristic function of a measurable set. Similar idea also gives better constants in the Lieb-Thirring inequalities for the respective moments of eigenvalues.

1. Let H be a Schrödinger operator in $\mathbb{R}^n$

$$H = -\Delta - V. \tag{1}$$

In what follows we can assume without loss of generality that the potential V is non-negative, $V \geq 0$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \leq 0$ be the non-positive eigenvalues of H. We shall study the quantities

$$S_{\gamma,n}(V) = \sum_j |\lambda_j|^\gamma, \qquad \gamma \geq 0. \tag{2}$$

It is known (see [8]) that there exist universal constants $L_{\gamma,n}$ such that

$$S_{\gamma,n}(V) \leq L_{\gamma,n} \int V^{\frac{n}{2}+\gamma} dx, \tag{3}$$

where $\gamma > 1/2$ $(n = 1)$, $\gamma > 0$ $(n = 2)$ and $\gamma \geq 0$ $(n \geq 3)$.

Let $N(\mu,V)$ be the number of bound states (non-positive eigenvalues) of the operator (1) not exceeding $-\mu$, $\mu \geq 0$. It is clear that $S_{0,n}(V) = N(0,V)$. Lieb and Thirring [10] proved the inequality (3) under the condition $\gamma > \max(0,1 - n/2)$ and obtained some estimates on the constants $L_{\gamma,n}$ (see also [8], [12], [4] for related papers). In the case $\gamma = 0$, $n \geq 3$ the inequality (3) was first obtained by Rozenblum [14]. Then independently of Rozenblum and of each other it was proved by Cwikel [3] and Lieb [6]. Later it was reproved again in [11]. All these proofs are different and give different estimates for the constants $L_{0,n}$. The best estimate is due to Lieb and all the attempts to improve it were unsuccessful.

2. In this paper we give a simple proof of the Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities for Schrödinger operators whose potentials equal the characteristic function of sets of the finite Lebesgue measure. Thus we study the eigenvalue problem for the Schrödinger operator in $\mathbb{R}^n$

$$H = -\Delta - \lambda\chi_\Omega, \qquad \lambda > 0, \tag{4}$$

where χ_Ω is the characteristic function of a measurable set $\Omega \subset \mathbb{R}^n$. Denote by v_n the volume of the unit ball $\mathbb{B}^n$.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a finite measure set, $|\Omega| \leq \infty$. Then

$$N(0,\lambda V) \leq M_{0,n} \lambda^{n/2} |\Omega|,$$

where

$$M_{0,n} = (2\pi)^{-n} v_{n-1} \frac{n^{n/2}}{(n-2)^{n/2}}.$$

Remark 1. If $n = 3$ we have $0.0780 < M_{0,3} = \frac{1}{8\pi^3} \frac{4\pi}{3} 3^{3/2} = \frac{\sqrt{3}}{2\pi^2} \approx 0.0877 < 0.1156$ which gives a better upper bound compared with [6].

Remark 2. Let $L^{cl}_{0,n} = (2\pi)^{-n} v_n$ be the constant appearing in the classical Weyl asymptotic formula for the number of negative eigenvalues of the operator (4) when $\lambda \to \infty$. Then

$$\lim_{n\to\infty} (L^{cl}_{0,n})^{-1} M_{0,n} = e.$$

Denote by λ^D_k the eigenvalues of the Dirichlet boundary value problem

$$-\Delta u = \lambda u, \quad u \in L_2(\Omega),$$

$$u|_{\partial\Omega} = 0.$$

Let $N^D_\Omega(\lambda)$ now equal the number of eigenvalues λ^D_k not exceeding λ. Applying Theorem 1 combined with the well-known variational principle (see [13]) we immediately obtain the following statement.

Corollary 2. If $|\Omega| < \infty$ and $n \geq 3$ then

$$N^D_\Omega(\lambda) \leq M_{0,n} \lambda^{n/2} |\Omega|. \tag{5}$$

Remark 3. Li and Yau [11] proved that for $n \geq 1$

$$\sum_{k=1}^{N} \lambda^D_k \geq \frac{n}{n+2} \frac{N^{\frac{n+2}{n}}}{(L^{cl}_{0,n}|\Omega|)^{2/n}}.$$

Since $N\lambda^D_N \geq \sum_{k=1}^N \lambda^D_k$ this implies that

$$N^D_\Omega(\lambda) \leq C_n \lambda^{n/2} |\Omega|, \tag{6}$$

where $C_n = L^{cl}_{0,n} \frac{(n+2)^{n/2}}{n^{n/2}}$. The last estimate is sharper than (5). However $\lim_{n\to\infty} M_{0,n} C_n^{-1} = 1$.

Proof of Theorem 1. Let $n(\alpha)$, $\alpha > 0$, be the number of eigenvalues of the operator

$$K = \lambda\chi_\Omega(-\Delta)^{-1}\chi_\Omega$$

which are greater or equal than α. Applying the Birman-Schwinger principle [1], [15] we obtain that $N(0,\lambda V) = n(1)$, and this therefore reduces the proof of Theorem 1 to the estimate of $n(1)$.

Now let $\varphi_a(t) = (t-a)_+$, $0 < a < 1$, $t \in \mathbb{R}^1$, be a convex function such that $\varphi_a(t) = t - a$ if $t > a$ and $\varphi_a(t) = 0$ if $t \le a$. Using the Berezin - Lieb trace inequality (see [2], [7] and also [5]) we have

$$n(1) \le \frac{1}{1-a}\mathrm{Tr}\,\varphi_a(\lambda\chi_\Omega(-\Delta)^{-1}\chi_\Omega) \le \frac{1}{1-a}\mathrm{Tr}\left(\chi_\Omega\varphi_a(\lambda(-\Delta)^{-1})\chi_\Omega\right). \qquad (7)$$

This implies that

$$\begin{aligned} n(1) &\le \frac{|\Omega|}{(1-a)(2\pi)^n}\int(\lambda|\xi|^{-2}-a)_+\,d\xi \\ &= \frac{\lambda^{n/2}a^{1-n/2}|\Omega|}{(1-a)(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^1 (r^{n-3}-r^{n-1})\,dr\,d\theta \\ &= \lambda^{n/2}\frac{a^{1-n/2}|\Omega|}{(1-a)(2\pi)^n}v_n\frac{2}{(n-2)}. \end{aligned}$$

The minimum value of the function $u(a) = (1-a)^{-1}a^{1-n/2}$ is reached at the point $a = 1 - 2/n$, and so we finally obtain

$$\begin{aligned} N(\lambda,0) &= n(1) \le \lambda^{n/2}\frac{(1-2/n)^{1-n/2}|\Omega|}{(1-(1-2/n))(2\pi)^n}v_n\frac{2}{(n-2)} \\ &= (2\pi)^{-n}\frac{n^{n/2}}{(n-2)^{n/2}}v_n\lambda^{n/2}|\Omega|. \end{aligned}$$

The proof is complete.

3. Using a similar approach we can find an upper bound on the quantity $S_{\gamma,n}(V)$, $\gamma > 0$, introduced in (2) when $V = \lambda\chi_\Omega$.

Theorem 3. If $|\Omega| < \infty$ and $n/2 + \gamma - 1 > 0$, $\gamma > 0$, then

$$S_{\gamma,n}(\lambda V) \le M_{\gamma,n}\lambda^{n/2+\gamma}|\Omega|,$$

where

$$M_{\gamma,n} = (2\pi)^{-n}v_n\Big(\frac{n/2+\gamma}{n/2+\gamma-1}\Big)^{n/2+\gamma}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{n}{2}+1)}.$$

Proof. It is clear that

$$S_{\gamma,n}(\lambda V) = \gamma \int_0^\infty \mu^{\gamma-1} N(\mu, \lambda\chi_\Omega)\, d\mu.$$

Therefore by analogy with (7) we obtain for any $0 \le a \le 1$ that

$$S_{\gamma,n}(\lambda V) \le \frac{\gamma}{1-a} \int \mu^{\gamma-1} \mathrm{Tr}\left(\chi_\Omega \varphi_a(\lambda(-\Delta+\mu)^{-1})\chi_\Omega\right) d\mu. \tag{8}$$

The trace in the right hand side of (8) can be calculated explicitly and we have

$$\begin{aligned} S_{\gamma,n}(\lambda V) &\le \frac{|\Omega|\gamma}{(2\pi)^n(1-a)} \int_0^\infty \int \mu^{\gamma-1}(\lambda(|\xi|^2+\mu)^{-1} - a)_+ \, d\xi\, d\mu \\ &= \frac{\lambda^{n/2+\gamma}|\Omega|}{(2\pi)^n} \frac{a^{1-n/2-\gamma}}{(1-a)} v_n \frac{\Gamma(n/2+1)\Gamma(\gamma+1)}{(\gamma+n/2-1)\Gamma(n/2+\gamma)}. \end{aligned}$$

Minimizing $a^{1-n/2-\gamma}(1-a)^{-1}$ we obtain that $a = 1-(n/2+\gamma)^{-1}$ and this completes the proof of the theorem.

Remark 4. If $n=3$ and $\gamma=1$ then $M_{1,3} \approx 0.02422$, which is better than 0.04030 obtained in [8]. However this still does not prove the conjecture (see [8], page 476) that $L_{1,3} < L^{cl}_{0,3} = 0.01689$. The inequality $M_{1,n} < L^{cl}_{0,n}$ becomes true only begining from $n \ge 5$.

Remark 5. By analogy with Remark 2 we obtain that

$$\lim_{n\to\infty} (L^{cl}_{\gamma,n})^{-1} M_{\gamma,n} = e,$$

where $L^{cl}_{\gamma,n}$ is the constant appearing in the classical Weyl asymptotic formula for the value $S_{\gamma,n}(\lambda\chi_\Omega)$ when $\lambda \to \infty$.

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