TCC - Problems Sheet 2

1. Follow the same steps as in the proof of the Sobolev Theorem prove that for any $0 < s < d/2$ there is a constant $\tilde{S}_{d,s} > 0$ such that

$$\int_{\mathbb{R}^d} |\xi|^{2s} \, |\widehat{u}|^2 \, d\xi \geq \tilde{S}_{d,s} \left(\int_{\mathbb{R}^d} |u|^{2d/(d-2s)} \, dx \right)^{(d-2s)/d}.$$

2. Show that there is a constant $C > 0$ such that for all $\infty > q \geq 4$ and all $u \in H^1(\mathbb{R}^2)$,

$$\left(\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \right)^{(q-2)/q} \left(\int_{\mathbb{R}^2} |u|^2 \, dx \right)^{2/q} \geq C \, q^{-1+2/q} \left(\int_{\mathbb{R}^2} |\nabla u|^q \, dx \right)^{2/q}.$$

3. Show that if $\rho \neq 1$, then

$$\int_0^\infty \frac{|u(x)|^2}{x^2} \, x^\rho \, dx \leq \left(\frac{2}{\rho - 1} \right)^2 \int_0^\infty |u'(x)|^2 \, x^\rho \, dx,$$

where $u \in H^1(\mathbb{R}_+)$ if $r > 1$ and that $u \in H_0^1(\mathbb{R}_+)$ if $r < 1$.

4. Find an example showing that the ‘domain monotonicity’ fails for the Neumann Laplacian.