TCC - Problems Sheet 3

1. Prove the Pólya inequality for the Neumann problem. Namely, Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, which is tiling. Then for any $\lambda > 0$

$$N(\lambda, -\Delta_\Omega^N) \geq L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}.$$

2. Assume that the spectrum of $-\Delta_\Omega^N$ on an open $\Omega \subset \mathbb{R}^d$ of finite measure is discrete. Let μ_k be the non-decreasing sequence of eigenvalues (counting with multiplicities). Prove

$$\sum_k (\Lambda - \mu_k)_+ \geq (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ \, d\xi.$$

3. Let φ defined on $[0, \infty)$ is convex and let $\lim_{x\to\infty} \varphi(x) = 0$. Show that the second distributional derivative of φ is a positive measure.