

TCC - Problems Sheet 4

1. Find spectrum of the multi-dimensional harmonic oscillator

$$\mathcal{H} = -\Delta + \sum_{k=1}^d \omega_k^2 x_k^2, \quad \text{in } L^2(\mathbb{R}^d).$$

2. Find spectrum of Grushin's sub-elliptic operator in $L^2(\mathbb{R}^3)$

$$\mathcal{H} = -\Delta_z + 4|z|^2 \partial_t^2, \quad z \in \mathbb{R}^2, t \in \mathbb{R}.$$

3. Let

$$Q = \frac{d}{dx} + x - \frac{1}{x} \quad \text{and} \quad Q^* = -\frac{d}{dx} + x - \frac{1}{x}$$

Show that

$$0 \leq Q^*Q = -\frac{d^2}{dx^2} + x^2 - 3.$$

Does it imply that $-\frac{d^2}{dx^2} + x^2 \geq 3$?

- 4.

2. Prove that the n^{th} eigenfunction φ_n of the operator

$$\mathcal{H}_\nu = -\frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{\cosh^2 x}$$

corresponding to the eigenvalue $-(\nu - n + 1)^2$, $n = 1, 2, \dots, [\nu]$ equals

$$\begin{aligned} & \varphi_n(x) \\ &= \cosh^{-\nu} \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \frac{\Gamma(n) \Gamma(\nu - n + 2)}{4^k \Gamma(n - 2k) \Gamma(\nu + k - n + 2) \Gamma(k + 1)} \sinh^{n-2k-1}(x). \end{aligned}$$