TCC - Problems Sheet 5

1. Let $\mathcal{H}$ be a 1D Schrödinger operator

$$\mathcal{H} = -\frac{d^2}{dx^2} - V(x), \qquad \text{in } L^2(\mathbb{R}),$$

where $V \to 0$ as $|x| \to \infty$ and $V \geq 0$ and let $(-\lambda_1, \psi_1)$ be its lowest eigenvalue and its respective eigenfunction. Show that $\psi_1(x) \neq 0$, $x \in \mathbb{R}$.

2. Let $n > 0$ and let

$$D_\nu = \frac{d}{dx} + \nu \tanh x, \qquad \text{in} \quad L^2(\mathbb{R}).$$

Then we have from Lecture 5 that

$$D_\nu^* D_\nu - \nu = -\frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{\cosh^2 x}.$$

For a $V > 0$ we introduce the eigenvalue λ_1 and the respective eigenfunction ψ_1 of the operator

$$H_\nu \psi_1 = -\frac{d^2}{dx^2}\psi_1 - \frac{\nu(\nu+1)}{\cosh^2 x}\psi_1 - V\psi_1 = -\lambda_1 \psi_1.$$

Then

$$D_\nu^* D_\nu \psi_1 - V\psi_1 = (-\lambda_1 + \nu)\psi_1.$$

Define

$$f_1 = \frac{D_\nu \psi_1}{\psi_1} \qquad \text{and} \qquad Q_1(\nu) = D_\nu - f_1.$$

Compute f_1' and also

$$Q_1^*(\nu) Q_1(\nu) \qquad \text{and} \qquad Q_1(\nu) Q_1^*(\nu).$$

3. Similarly to Problem 2 consider

$$D = \frac{d}{dx} + x, \qquad \text{in} \quad L^2(\mathbb{R}).$$

Then

$$D^* D + 1 = -\frac{d^2}{dx^2} + x^2.$$

For a $V > 0$ we introduce the eigenvalue λ_1 and the respective eigenfunction ψ_1 of the operator

$$H_\nu \psi_1 = -\frac{d^2}{dx^2}\psi_1 + x^2\psi_1 - V\psi_1 = -\lambda_1 \psi_1.$$

Then

$$D^* D\,\psi_1 - V\psi_1 = (-\lambda_1 - 1)\psi_1.$$

Define

$$f_1 = \frac{D\psi_1}{\psi_1} \qquad \text{and} \qquad Q_1 = D - f_1.$$

Compute f_1' and also

$$Q_1^* Q_1 \qquad \text{and} \qquad Q_1 Q_1^*.$$