TCC - Problems Sheet 6

1. Show that the inequality for negative eigenvalue λ of the Schröodinger operator $-\Delta - V$ in $L^2(\mathbb{R}^d)$

$$\lambda \geq -\left(L^1_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+d/2}\, dx\right)^{1/\gamma},$$

with

$$L^1_{\gamma,d} = \frac{\gamma^\gamma (d/2)^{d/2}}{(\gamma + d/2)^{\gamma+d/2}}\, S_{q,d}^{-\gamma-d/2}$$

is optimal.

2. Prove that if $V \in L^1(\mathbb{R}^2)$ and $\int V\, dx > 0$ then the operator $\mathcal{H} = -\Delta - V$ in $L^2(\mathbb{R}^2)$ has at least one negative eigenvalue.

3. Show that if $\int_{\mathbb{R}^d} V\, |x|^{-d+2}\, dx > 0$, then

$$N\left(0, -\Delta - \frac{(d-2)^2}{4|x|^2} - V\right) \geq 1.$$

4. Prove that for any compact self-adjoint operator A with eigenvalues $\{\mu_n\}$

$$|||A|||_N = \sum_{n=1}^{N} |\mu_n(A)|$$

is a norm for any $N \in \mathbb{N}$.

5. Show that the inequality

$$\sum_k \sqrt{\lambda_k} \leq \frac{1}{2} \int_{\mathbb{R}} V_+\, dx$$

is sharp.

Hint: Choose $V(x) = \delta(x)$.