THE ELEMENTARY OBSTRUCTION AND HOMOGENEOUS SPACES

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Abstract

Let k be a field of characteristic zero, and let $\overline{k}$ be an algebraic closure of k. For a geometrically integral variety X over k, we write $\overline{k}(X)$ for the function field of $\overline{X} = X \times_k \overline{k}$. If X has a smooth k-point, the natural embedding of multiplicative groups $\overline{k}^* \hookrightarrow \overline{k}(X)^*$ admits a Galois-equivariant retraction.

In the first part of this article, equivalent conditions to the existence of such a retraction are given over local and then over global fields. Those conditions are expressed in terms of the Brauer group of X.

In the second part of the article, we restrict attention to varieties that are homogeneous spaces of connected but otherwise arbitrary algebraic groups, with connected geometric stabilizers. For k local or global, and for such a variety X, in many situations but not all, the existence of a Galois-equivariant retraction to $\overline{k}^* \hookrightarrow \overline{k}(X)^*$ ensures the existence of a k-rational point on X. For homogeneous spaces of linear algebraic groups, the technique also handles the case where k is the function field of a complex surface.

Résumé

Soient k un corps de caractéristique nulle et $\overline{k}$ une clôture algébrique de k. Pour une k-variété X géométriquement intègre, on note $\overline{k}(X)$ le corps des fonctions de $\overline{X} = X \times_k \overline{k}$. Si X possède un k-point lisse, le plongement naturel de groupes multiplicatifs $\overline{k}^* \hookrightarrow \overline{k}(X)^*$ admet une rétraction équivariante pour l'action du groupe de Galois de $\overline{k}$ sur k.

Dans la première partie de l'article, sur les corps locaux puis sur les corps globaux, on donne des conditions équivalentes à l'existence d'une telle rétraction équivariante. Ces conditions s'expriment en terme du groupe de Brauer de la variété X.

Dans la seconde partie de l'article, on considère le cas des espaces homogènes de groupes algébriques connexes, non nécessairement linéaires, avec groupes d'isotropie

DUKE MATHEMATICAL JOURNAL
Vol. 141, No. 2, © 2008
Received 6 November 2006. Revision received 23 April 2007.
2000 *Mathematics Subject Classification*. Primary 14G05, 11G99, 12G05; Secondary 11E72, 14F22, 14K15, 14M17, 20G99.
Borovoi's work partially supported by the Hermann Minkowski Center for Geometry and by the Ministry of Science, Culture, and Sport, Israel, and the Russian Foundation for Basic Research grant 3-3578.
Colliot-Thélène's work partially supported by the Clay Mathematics Institute.

géométriques connexes. Pour k local ou global, pour un tel espace homogène X, dans beaucoup de cas mais pas dans tous, l'existence d'une rétraction équivariante à $\overline{k}^* \hookrightarrow \overline{k}(X)^*$ *implique l'existence d'un point k-rationnel sur X. Pour les espaces homogènes de groupes linéaires, la technique permet aussi de traiter le cas où k est un corps de fonctions de deux variables sur les complexes.*

Contents

1. Introduction

Among the many obstructions to the existence of rational points, one is particularly remarkable due to the simplicity of its construction.

Let k be a field of characteristic zero, let $\overline{k}$ be an algebraic closure of k, and let $\mathfrak{g}$ be the Galois group of $\overline{k}$ over k. For a geometrically integral variety X over k, we write $\overline{k}(X)$ for the function field of $\overline{X} = X \times_k \overline{k}$. The elementary obstruction, defined and studied in [11], is the class $ob(X) \in \mathrm{Ext}^1_{\mathfrak{g}}(\overline{k}(X)^*/\overline{k}^*, \overline{k}^*)$ of the extension of Galois modules

$$1 \to \overline{k}^* \to \overline{k}(X)^* \to \overline{k}(X)^*/\overline{k}^* \to 1. \tag{1}$$

If X and Y are geometrically integral k-varieties, and if there exists a dominant rational map f from X to Y, then $ob(X) = 0$ implies that $ob(Y) = 0$. In particular, the vanishing of $ob(X)$ is a birational invariant of X. As pointed out by Wittenberg [49, Lem. 3.1.2], there is a more general result: if there exists a rational map from a geometrically integral variety X to a smooth, geometrically integral k-variety Y, then $ob(X) = 0$ implies that $ob(Y) = 0$.

As a special case, if X has a smooth k-point, the extension (1) is split, so that $ob(X) = 0$ (see [11, Prop. 2.2.2]).

Thus we are confronted with the following natural question: for which fields k and k-varieties X is $ob(X)$ the only obstruction to the existence of k-points on X?

In the first part of this article, we consider arbitrary smooth, geometrically integral varieties. After recalling some general facts about the elementary obstruction, we turn to local and global fields. For such fields, we relate the elementary obstruction to obstructions coming from the Brauer group, as follows.

(i) If k is local (e.g., a p-adic field or the field of real numbers), $ob(X) = 0$ if and only if the natural map $\operatorname{Br} k \to \operatorname{Br} k(X)$ is injective (see Theorems 2.5 and 2.6 for more general statements).

(ii) If k is a number field, if $ob(X) = 0$, and if X has points in all completions of k, then any adèle of X is orthogonal to the subgroup of the Brauer group of X consisting of "algebraic" elements that are everywhere locally constant (Th. 2.13).

In the second part of this article, we explore the elementary obstruction $ob(X)$, when X is a homogeneous space of a connected algebraic group G, not necessarily linear. Most results require the assumption that the stabilizers of $\bar{k}$-points of X are connected. Under this assumption, we prove the following results.

(iii) If k is a p-adic field, we show that $ob(X) = 0$ implies the existence of a rational point (Th. 3.3). This actually holds as long as the Brauer group of k injects into the Brauer group of the function field of X (Cor. 3.4). The case of homogeneous spaces of abelian varieties was already known (see Lichtenbaum [26], van Hamel [46]).

(iv) If k is a "good" field of cohomological dimension at most 2, and if the group G is linear, then the hypothesis $ob(X) = 0$ implies the existence of a rational point (Th. 3.8). This result covers the case of p-adic fields—already handled in (iii)—and of totally imaginary number fields. Thanks to a theorem of de Jong [15], it also applies to function fields in two variables over an algebraically closed field, provided that G has no factor of type E_8.

(v) If k is a number field, if the group G is linear, and if X has points in the real completions of k and $ob(X) = 0$, then X has a rational point (Th. 3.10).

(vi) If k is a totally imaginary number field, and if G is an arbitrary connected algebraic group, assuming finiteness of the Tate-Shafarevich group of the maximal abelian variety quotient of G, we prove that $ob(X) = 0$ implies that X has a rational point (Th. 3.14). A key ingredient is a recent result of Harari and Szamuely [25] on principal homogeneous spaces of commutative algebraic groups. Their theorem also holds when k has real completions.

(vii) In the general case of arbitrary connected groups, we found, somewhat to our surprise, a principal homogeneous space $X/\mathbf{Q}$ of a noncommutative group G with $ob(X) = 0$ and with points everywhere locally, but without $\mathbf{Q}$-points (Prop. 3.16). Using either Theorem 2.13 or an easy direct argument, one sees that the Brauer-Manin obstruction attached to the subgroup $Б(X) \subset \operatorname{Br}_1 X$ of everywhere locally constant classes is trivial. Thus we obtain a negative answer to the following question raised in [41, Ques. 1, p. 133]: is the Brauer-Manin obstruction attached to $Б(X)$ the only obstruction to the Hasse principle for torsors of arbitrary connected algebraic groups? This phenomenon is due to a combination of three factors: the formal reality of the ground field, the noncommutativity of G, and its nonlinearity.

The example in (vii) can be accounted for by the Brauer-Manin obstruction attached to the group $\mathrm{Br}_1 X^c$, where X^c denotes a smooth compactification of the torsor X. This is a special instance of a result of Harari [24]. In the appendix, building upon [24] and the techniques used in the present article, we extend Harari's result to homogeneous spaces of any connected algebraic group G, assuming that the geometric stabilizers are connected. As in [24] and earlier work on the subject, the result here is conditional on the finiteness of the Tate-Shafarevich group of the maximal abelian variety quotient of G.

In the case of a linear algebraic group G, the recurring assumption that the geometric stabilizers are connected can be somewhat relaxed (Ths. 3.5 and A.5), but some condition must definitely be imposed, as shown by an example of Florence [17].

The starting point of our work was the following result of van Hamel: for a principal homogeneous space X of a connected linear k-group G over a p-adic field k, the elementary obstruction is the only obstruction to the existence of a k-rational point on X.

2. Elementary obstruction

2.1. Preliminaries

Let k be a field of characteristic zero, let $\overline{k}$ be an algebraic closure of k, and let $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$. If X is a k-variety, we let $\overline{X} = X \times_k \overline{k}$. If X is integral, we denote by $k(X)$ the function field of X. If X is geometrically integral, we denote by $\overline{k}(X)$ the function field of $\overline{X}$. We let $\mathrm{Div}\, X$ denote the group of Cartier divisors on X, and we let $\mathrm{Pic}\, X$ denote the Picard group $\mathrm{H}^1_{\mathrm{Zar}}(X, \mathbf{G}_m) = \mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ of X. By $\mathrm{Br}\, X$, we denote the cohomological Brauer-Grothendieck group $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$, and by $\mathrm{Br}_1 X$, we denote the kernel of the natural map $\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}$. If M is a continuous discrete $\mathfrak{g}$-module, we write $\mathrm{H}^i(k, M)$ for the Galois cohomology groups.

When $\overline{k}^* = \overline{k}[X]^*$, the Hochschild-Serre spectral sequence

$$E_2^{pq} = \mathrm{H}^p\big(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbf{G}_m)\big) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbf{G}_m)$$

gives rise to the well-known exact sequence

$$0 \to \mathrm{Pic}\, X \to (\mathrm{Pic}\, \overline{X})^{\mathfrak{g}} \to \mathrm{Br}\, k \to \mathrm{Br}_1 X \xrightarrow{r} \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X}), \tag{2}$$

where the map $\mathrm{Br}_1 X \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{X})$ is onto if X has a k-point or if k is a local or global field.

Recall that if A and B are continuous discrete $\mathfrak{g}$-modules, then $\mathrm{Ext}^n_{\mathfrak{g}}(A, B)$ is defined as the derived functor of $\mathrm{Hom}_{\mathfrak{g}}(A, B)$ in the second variable. In particular, there are long exact sequences in either variable, and the elements of $\mathrm{Ext}^n_{\mathfrak{g}}(A, B)$ classify equivalence classes of n-extensions of continuous discrete Galois modules (for further specifics on this, see [31, Chap. III, Sec. 1, Rem. 1.6]).

Let X be a smooth, quasi-projective, and geometrically integral variety over k. Then Cartier divisors coincide with Weil divisors, which implies that $\mathrm{Div}\,\overline{X}$ is a permutation $\mathfrak{g}$-module. We have the following natural 2-extension of continuous discrete $\mathfrak{g}$-modules:

$$1 \to \bar{k}[X]^* \to \bar{k}(X)^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0.$$

When $\overline{k}^* = \bar{k}[X]^*$, this reads

$$1 \to \overline{k}^* \to \bar{k}(X)^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0. \tag{3}$$

Under the assumption that $\overline{k}^* = \bar{k}[X]^*$, write $e(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*)$ for the corresponding class.[†] Much is known about the classes $ob(X)$ and $e(X)$ (see [11, Sec. 2], [41, Chap. 2]). Clearly, $e(X)$ is the cup product of

$$1 \to \bar{k}(X)^*/\overline{k}^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0$$

with the class $ob(X)$. For further reference, we list here some of the known properties of these classes.

LEMMA 2.1

(i) *The class $ob(X)$ lies in the kernel of the natural map*

$$\mathrm{Ext}^1_{\mathfrak{g}}\big(\bar{k}(X)^*/\overline{k}^*, \overline{k}^*\big) \to \mathrm{Ext}^1_{\mathfrak{g}}\big(\bar{k}(X)^*/\overline{k}^*, \bar{k}(X)^*\big).$$

(ii) *If there exists a zero-cycle of degree 1 on X, then $ob(X) = 0$.*

(iii) *If $ob(X) = 0$, then for a k-group of multiplicative type S and $i = 0, 1, 2$, the natural maps $\mathrm{H}^i(k, S) \to \mathrm{H}^i(k(X), S)$ are injective. In particular, the map $\mathrm{Br}\,k \to \mathrm{Br}\,k(X)$ is injective, and so is the map $\mathrm{Br}\,k \to \mathrm{Br}\,X$.*

(iv) *If X is k-birational to a homogeneous space of a k-torus, then $ob(X) = 0$ if and only if $X(k) \neq \emptyset$.*

Proof

(i) This statement is obvious.

(ii) See [11, Prop. 2.2.2], [41, Th. 2.3.4].

(iii) See [11, Prop. 2.2.5].

(iv) We may assume that X is a k-torsor of a k-torus (see [3, proof of Prop. 3.3]). If $ob(X) = 0$, then $\overline{k}^*$ is a direct summand in $\bar{k}(X)^*$; hence it is also a direct summand in $\bar{k}[X]^*$. Now, it follows from [36, (6.7.3), (6.7.4)] that X is a trivial torsor (i.e., X has a k-point). □

[†]This definition of $e(X)$ differs from that in [41] by -1.

LEMMA 2.2
Assume that $\overline{k}^* = \overline{k}[X]^*$.

(i) *We have* $ob(X) = 0$ *if and only if* $e(X) = 0$.

(ii) *The class* $e(X)$ *lies in the kernel of the natural map*

$$\mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*) \to \mathrm{Ext}^2_{\mathfrak{g}}\big(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*\big).$$

(iii) *The map* $(\mathrm{Pic}\,\overline{X})^{\mathfrak{g}} \to \mathrm{Br}\,k$ *in (2) is the Yoneda product with* $e(X)$ *(up to sign).*

(iv) *If* $\mathrm{Pic}\,\overline{X}$ *is finitely generated and free as an abelian group, and if* S *denotes the* k*-torus with character group* $\mathrm{Pic}\,\overline{X}$*, then* $ob(X) = 0$ *if and only if* $\mathrm{H}^2(k, S)$ *injects into* $\mathrm{H}^2(k(X), S)$.

(v) *If* $\mathrm{Pic}\,\overline{X} = \mathbf{Z}$*, then* $ob(X) = 0$ *if and only if the map* $\mathrm{Br}\,k \to \mathrm{Br}\,k(X)$ *is injective.*

(vi) *If* $\mathrm{Pic}\,\overline{X}$ *is finitely generated and is a direct factor of a permutation* $\mathfrak{g}$*-module, then* $ob(X) = 0$ *if and only if, for any finite field extension* K/k*, the map* $\mathrm{Br}\,K \to \mathrm{Br}\,K(X)$ *is injective.*

(vii) *If* $\mathrm{Pic}\,\overline{X} = 0$*, then* $ob(X) = 0$.

(viii) *If* X *is a principal homogeneous space of a semisimple simply connected group, then* $ob(X) = 0$.

(ix) *If* $X \subset \mathbf{P}^n_k$ *is a smooth, projective hypersurface, and if* $n \geq 4$*, then* $ob(X) = 0$.

Proof

(i) See [11, Prop. 2.2.4], [41, Th. 2.3.4].

(ii) This assertion follows from Lemma 2.1(i).

(iii) See [11, Lem. 1.A.4], [42, Prop. 1.1].

(iv) The direct implication follows from Lemma 2.1(iii). For the converse, observe that the natural map $\mathrm{H}^2(k, S) \to \mathrm{H}^2(k(X), S)$ factors through

$$\mathrm{H}^2\big(k, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*)\big) \to \mathrm{H}^2\big(k, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*)\big). \tag{4}$$

Since the $\mathfrak{g}$-module $\mathrm{Pic}\,\overline{X}$ is finitely generated, we have the spectral sequence

$$\mathrm{H}^p\big(k, \mathrm{Ext}^q_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*)\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathfrak{g}}\big(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*\big).$$

Since $\mathrm{Pic}\,\overline{X}$ is finitely generated and torsion-free, for any $q \geq 1$ we have $\mathrm{Ext}^q_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*) = 0$, so that the spectral sequence degenerates and gives an isomorphism $\mathrm{H}^2\big(k, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*)\big) = \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}(X)^*)$. This and a similar argument for $\mathrm{H}^2(k, \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*))$ identify (4) with the map in (ii). Now, our statement follows from (i) and (ii).

(v) This is a special case of (iv).

(vi) Assume that $ob(X) = 0$. Let K/k be a finite field extension. Applying Lemma 2.1(iii) to the k-torus $S = R_{K/k}\mathbf{G}_m$ and using Shapiro's lemma, one finds that $\operatorname{Br} K \to \operatorname{Br} K(X)$ is injective. One can also directly argue that $ob(X) = 0$ implies that $ob(X \times_k K) = 0$. Assume now that $\operatorname{Pic} \overline{X}$ is finitely generated and is a direct factor of a permutation $\mathfrak{g}$-module $\oplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i]$, where $\mathfrak{g}_i = \operatorname{Gal}(\overline{k}/K_i)$, with each $K_i \subset \overline{k}$ a finite field extension of k. Let S, respectively, P, be the k-torus whose character group is $\operatorname{Pic} \overline{X}$, respectively, $\oplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i]$. There exist a k-torus S_1 and an isomorphism of k-tori $S \times_k S_1 \simeq P$. Let us assume that for each K_i/k, the natural map $\operatorname{Br} K_i \to \operatorname{Br} K_i(X)$ is injective. By Shapiro's lemma, this is equivalent to assuming the injectivity of the natural map $\mathrm{H}^2(k, P) \to \mathrm{H}^2(k(X), P)$. This, in turn, implies the injectivity of $\mathrm{H}^2(k, S) \to \mathrm{H}^2(k(X), S)$. From (iv), we conclude that $ob(X) = 0$.

(vii) Given (3), this is an application of (i) (cf. [11, Rem. 2.2.7]).

(viii) This is a direct application of (vii).

(ix) For such a hypersurface, the restriction map $\mathbf{Z} = \operatorname{Pic} \mathbf{P}^n_k \to \operatorname{Pic} X$ is an isomorphism, and so it is over $\overline{k}$ (Max Noether's theorem). Let $U \subset X$ be the complement of a smooth hyperplane section defined over k. Then $\overline{k}^* = \overline{k}[U]^*$ and $\operatorname{Pic} \overline{U} = 0$. One may then apply (vii) to U. □

Remarks

(1) There exist higher Galois cohomological obstructions to the existence of rational points and, more generally, to the existence of a zero-cycle of degree 1. Let X be a smooth, geometrically integral k-variety, and let S be a k-group of multiplicative type, for instance, a finite $\mathfrak{g}$-module. If X has a zero-cycle of degree 1, then for any positive integer n, the restriction map $\mathrm{H}^n(k, S) \to \mathrm{H}^n(k(X), S)$ is injective: this is a consequence of the Bloch-Ogus theorem (see [5]).

(2) In [11, exem. 2.2.12], there is a sample of varieties over suitable fields satisfying $ob(X) = 0$ but lacking k-rational points. Simple examples with $ob(X) = 0$ are given by Lemma 2.2(viii) and (ix). Some of these examples can be explained by means of the higher Galois obstructions in Remark (1), whereas some others cannot (for more on this, see the remarks after Theorems 2.5 and 2.6).

(3) Let $k = \mathbf{C}((t))$. Let X/k be the curve of genus 1 defined by the homogeneous equation $x^3 + ty^3 + t^2z^3 = 0$. We obviously have $X(k) = \emptyset$. The Brauer group of k and of any finite extension of k vanishes. A general result of Wittenberg [49, Th. 3.4.1] then ensures that $ob(X) = 0$. Thus the absence of k-points on X is not detected by any of the above Galois cohomology arguments.

Questions

Let X be a geometrically integral k-variety. Let K/k be an arbitrary field extension.

(1) Assume that $ob(X) = 0$. Does the K-variety X_K satisfy $ob(X_K) = 0$? This is clear if $K \subset \overline{k}$.

(2) Assume that the K-variety X_K satisfies $ob(X_K) = 0$. If the extension K/k has a k-place, does the k-variety X satisfy $ob(X) = 0$?

We can answer the first question in a special case.

PROPOSITION 2.3
Let X/k be a smooth, projective, geometrically integral variety. Assume that the Picard variety of X is trivial. If $ob(X) = 0$, then for any field K containing k, we have $ob(X_K) = 0$.

Proof
Let $\overline{k} \subset \overline{K}$ be an inclusion of algebraic closures. Let $\mathfrak{G} = \mathrm{Gal}(\overline{K}/K)$, and let $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$. There is a natural map $\mathfrak{G} \to \mathfrak{g}$. Because the Picard variety of X is trivial, the abelian groups $\mathrm{Pic}\, X_{\overline{k}}$ and $\mathrm{Pic}\, X_{\overline{K}}$ are abelian groups of finite type, and the natural map $\mathrm{Pic}\, X_{\overline{k}} \to \mathrm{Pic}\, X_{\overline{K}}$ is a Galois equivariant isomorphism. (The Néron-Severi group does not change under extensions of algebraically closed ground fields.) There is an equivariant commutative diagram of 2-extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{k}^* & \longrightarrow & \overline{k}(X)^* & \longrightarrow & \mathrm{Div}\, X_{\overline{k}} & \longrightarrow & \mathrm{Pic}\, X_{\overline{k}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \simeq & & \\ 1 & \longrightarrow & \overline{K}^* & \longrightarrow & \overline{K}(X)^* & \longrightarrow & \mathrm{Div}\, X_{\overline{K}} & \longrightarrow & \mathrm{Pic}\, X_{\overline{K}} & \longrightarrow & 0 \end{array} \tag{5}$$

If $ob(X) = 0$, then the top 2-extension is trivial (see Lem. 2.2(i)). This implies that the bottom 2-extension is trivial; that is, $ob(X_K) = 0$. □

Other cases where Question (1) can be answered positively are handled in Sections 2.2 and 2.3 (for further results, see [49]).†

Let X/k be a smooth, projective, geometrically integral variety. Let J/k be the Picard variety of X. Let $\mathrm{NS}\, \overline{X}$ be the Néron-Severi group of $\overline{X}$. From the exact sequence of $\mathfrak{g}$-modules

$$0 \to J(\overline{k}) \to \mathrm{Pic}\, \overline{X} \to \mathrm{NS}\, \overline{X} \to 0, \tag{6}$$

we deduce the following diagram, in which the vertical sequences are exact:

$$\begin{array}{ccccc} \mathrm{H}^1(k, J) & \times & \mathrm{Ext}^1_{\mathfrak{g}}\big(J(\overline{k}), \overline{k}^*\big) & \longrightarrow & \mathrm{Br}\, k \\ \uparrow & & \downarrow & & \| \\ (\mathrm{NS}\, \overline{X})^{\mathfrak{g}} & \times & \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\, \overline{X}, \overline{k}^*) & \longrightarrow & \mathrm{Br}\, k \\ \uparrow & & \downarrow & & \| \\ (\mathrm{Pic}\, \overline{X})^{\mathfrak{g}} & \times & \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\, \overline{X}, \overline{k}^*) & \longrightarrow & \mathrm{Br}\, k \\ \uparrow & & \downarrow & & \| \\ J(k) & \times & \mathrm{Ext}^2_{\mathfrak{g}}\big(J(\overline{k}), \overline{k}^*\big) & \longrightarrow & \mathrm{Br}\, k \end{array} \tag{7}$$

†Added in print: O. Wittenberg has just shown that the answer to Question (1), in general, is in the negative.

This diagram is commutative, except for the upper square, which is anticommutative (with the sign conventions of [27]). The middle and the lower squares are obvious, so we just need to explain the upper square. The associativity of the Yoneda product (see [27, Chap. III, Th. 5.3]) implies the commutativity of the upper square if the maps are the products with the class of (6). By [27, Chap. III, Th. 9.1], such is the left-hand vertical map, but the right-hand one differs from the Yoneda product by -1.

Let A denote the Albanese variety of X. The abelian varieties J and A are dual to each other. A choice of a $\overline{k}$-point on X defines the Albanese map $\overline{X} \to \overline{A}$ over $\overline{k}$, sending this point to zero. This map canonically descends to a morphism $X \to D$, where D is a k-torsor of A (cf. [41, Sec. 3.3]). Let $\delta(X) \in \mathrm{H}^1(k, A)$ be the class of D. This class does not depend on any choice. In the particular case where X is a k-torsor of an abelian variety, the map $X \to D$ is an isomorphism, so that $X(k) \neq \emptyset$ if and only if $\delta(X) = 0$.

The Barsotti-Weil isomorphism $A(\overline{k}) = \mathrm{Ext}^1_{\overline{k}-\mathrm{gps}}(J, \mathbf{G}_m)$ (see [38, Chap. VII, Sec. 3]) gives rise to natural isomorphisms (see [32, Lem. 3.1, p. 50]):

$$\mathrm{H}^n(k, A) = \mathrm{Ext}^{n+1}_{k-\mathrm{gps}}(J, \mathbf{G}_m), \tag{8}$$

where $k - \mathrm{gps}$ is the category of commutative algebraic groups over k, and n is a nonnegative integer. Here the Ext^n-groups are defined by means of equivalence classes of n-extensions.

Building upon these isomorphisms, one defines two Tate pairings.

The first Tate pairing,

$$\mathrm{H}^1(k, J) \times A(k) \to \mathrm{Br}\, k,$$

is defined by means of the composition of maps

$$A(k) = \mathrm{Ext}^1_{k-\mathrm{gps}}(J, \mathbf{G}_m) \to \mathrm{Ext}^1_{\mathfrak{g}}\big(J(\overline{k}), \overline{k}^*\big) \to \mathrm{Hom}\big(\mathrm{H}^1(k, J), \mathrm{Br}\, k\big),$$

where the first map is the isomorphism (8) for $n = 0$, the second map is the forgetful map, and the third map is the Yoneda pairing.

The second Tate pairing,

$$J(k) \times \mathrm{H}^1(k, A) \to \mathrm{Br}\, k,$$

is defined by means of the composition of maps

$$\mathrm{H}^1(k, A) = \mathrm{Ext}^2_{k-\mathrm{gps}}(J, \mathbf{G}_m) \to \mathrm{Ext}^2_{\mathfrak{g}}\big(J(\overline{k}), \overline{k}^*\big) \to \mathrm{Hom}\big(J(k), \mathrm{Br}\, k\big),$$

where the first map is the isomorphism (8) for $n = 1$, the second map is the forgetful map, and the third map is the Yoneda pairing.

(A legitimate question, which we need not address, is whether these two pairings coincide upon swapping A with J. As a referee pointed out, biextensions should help.)

The second Tate pairing fits into the commutative diagram

$$\begin{array}{ccccc} (\operatorname{Pic}\overline{X})^{\mathfrak{g}} & \times & \operatorname{Ext}^2_{\mathfrak{g}}(\operatorname{Pic}\overline{X},\overline{k}^*) & \longrightarrow & \operatorname{Br} k \\ \uparrow & & \downarrow & & \| \\ J(k) & \times & \operatorname{Ext}^2_{\mathfrak{g}}\big(J(\overline{k}),\overline{k}^*\big) & \longrightarrow & \operatorname{Br} k \\ \| & & \uparrow & & \| \\ J(k) & \times & \mathrm{H}^1(k, A) & \longrightarrow & \operatorname{Br} k \end{array} \tag{9}$$

where the top square comes from the diagram (7) (the pairing in the middle being the Yoneda pairing).

PROPOSITION 2.4 ([42, Prop. 2.1])
In this diagram, the image of $e(X) \in \operatorname{Ext}^2_{\mathfrak{g}}(\operatorname{Pic}\overline{X},\overline{k}^*)$ *in* $\operatorname{Ext}^2_{\mathfrak{g}}(J(\overline{k}),\overline{k}^*)$ *is equal to the image of* $\delta(X) \in \mathrm{H}^1(k, A)$ *in* $\operatorname{Ext}^2_{\mathfrak{g}}(J(\overline{k}),\overline{k}^*)$.

2.2. The Brauer group and the elementary obstruction over local fields

Let R be a Henselian, discrete, rank 1 valuation ring with finite residue field and field of fractions k of characteristic zero. We refer to such a field as a *Henselian local field* (for k of arbitrary characteristic, see [32, Chap. I.2, p. 43]). A Henselian local field is a p-adic field if and only if it is complete.

THEOREM 2.5
Let X be a geometrically integral variety over a Henselian local field k. Then $ob(X) = 0$ if and only if the natural map $\operatorname{Br} k \to \operatorname{Br} k(X)$ *is injective.*

Proof
Over any field, the assumption $ob(X) = 0$ implies that $\operatorname{Br} k \to \operatorname{Br} k(X)$ is injective (see Lemma 2.1(iii)).

Using resolution of singularities, we may assume that X is smooth and projective. Assume that $\operatorname{Br} k \to \operatorname{Br} k(X)$ is injective. This implies that $\operatorname{Br} k \to \operatorname{Br} X$ is injective, and hence the map $(\operatorname{Pic}\overline{X})^{\mathfrak{g}} \to \operatorname{Br} k$ in sequence (2) is zero. This map is the cup product with $e(X)$ (see Lem. 2.2(iii)); thus $e(X)$ is orthogonal to $(\operatorname{Pic}\overline{X})^{\mathfrak{g}}$ with respect to the Yoneda product.

Consider diagram (7). Now, $(\operatorname{Pic}\overline{X})^{\mathfrak{g}}$ is orthogonal to $e(X) \in \operatorname{Ext}^2_{\mathfrak{g}}(\operatorname{Pic}\overline{X},\overline{k}^*)$; thus the image of $e(X)$ in $\operatorname{Ext}^2_{\mathfrak{g}}(J(\overline{k}),\overline{k}^*)$ is orthogonal to $J(k)$. As recalled in Proposition 2.4, this image is equal to the image of $\delta(X)$ under the bottom right-hand vertical map in diagram (9). From that diagram, we conclude that $\delta(X) \in \mathrm{H}^1(k, A)$ is orthogonal to $J(k)$ under the second Tate pairing. By Tate's second duality theorem (see [32, Chap. I.3, Th. 3.2 (statement for α^2), Cor. 3.4 and Rem. 3.10, line 5 on p. 59]), this implies that $\delta(X) = 0$. Hence the image of $e(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*)$ in $\mathrm{Ext}^2_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*)$ is zero. Thus $e(X)$ is the image of some element $g(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*)$. This element is orthogonal to the image of $(\mathrm{Pic}\,\overline{X})^{\mathfrak{g}}$ in $(\mathrm{NS}\,\overline{X})^{\mathfrak{g}}$. Let $M \subset \mathrm{H}^1(k, J)$ be the image of $(\mathrm{NS}\,\overline{X})^{\mathfrak{g}}$. Since the abelian group $\mathrm{NS}\,\overline{X}$ is finitely generated, and $\mathrm{H}^1(k, J)$ is torsion, the abelian group M is finite. The cup product with $g(X)$ defines a map

$$(\mathrm{NS}\,\overline{X})^{\mathfrak{g}} \to \mathrm{Br}\,k = \mathbf{Q}/\mathbf{Z},$$

which induces a map $\nu : M \to \mathbf{Q}/\mathbf{Z}$. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group, the following natural homomorphism is surjective:

$$\mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}^1(k, J), \mathbf{Q}/\mathbf{Z}\big) \to \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}). \tag{10}$$

As explained above, the Barsotti-Weil isomorphism (8) $A(k) = \mathrm{Ext}^1_{k-\mathrm{gps}}(J, \mathbf{G}_m)$ and the forgetful map $\mathrm{Ext}^1_{k-\mathrm{gps}}(J, \mathbf{G}_m) \to \mathrm{Ext}^1_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*)$ give rise to the diagram

$$\begin{tikzcd}
\mathrm{H}^1(k, J) \times \arrow[d] & A(k) \arrow[r] \arrow[d] & \mathrm{Br}\,k \arrow[d, equal] \\
\mathrm{H}^1(k, J) \times & \mathrm{Ext}^1_{\mathfrak{g}}\big(J(\overline{k}), \overline{k}^*\big) \arrow[r] & \mathrm{Br}\,k
\end{tikzcd} \tag{11}$$

which is the definition of the upper-row pairing (see [32, Prop. 0.16, p. 14; Chap. I.3]): this is the first Tate pairing as defined at the end of Section 2.1. By Tate's first duality theorem over a Henselian local field (see [32, Chap. I.3, Th. 3.2 (statement for α^1), Cor. 3.4 and Rem. 3.10, line 5 on p. 59]), this pairing induces a perfect duality between the discrete group $\mathrm{H}^1(k, J)$ and the completion $A(k)\hat{}$ of $A(k)$ with respect to the natural topology on k. In particular, $A(k)$ is a dense subgroup of $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}^1(k, J), \mathbf{Q}/\mathbf{Z})$. By the surjectivity of (10), its image in $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ is also dense. Thus the image of $A(k)$ is the whole finite set $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$. Hence there exists an element of $A(k)$ which induces ν on M via the first Tate pairing. Let $\rho \in \mathrm{Ext}^1_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*)$ be its image. If one modifies $g(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*)$ by the image of ρ under the map $\mathrm{Ext}^1_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*) \to \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*)$, one obtains an element $g_1(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*)$ whose image in $\mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*)$ is still $e(X)$ but which is now orthogonal to $(\mathrm{NS}\,\overline{X})^{\mathfrak{g}}$ with respect to the cup-product pairing

$$(\mathrm{NS}\,\overline{X})^{\mathfrak{g}} \times \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*) \to \mathrm{Br}\,k.$$

The Néron-Severi group $\mathrm{NS}\,\overline{X}$ is a discrete Galois module of finite type. Over the Henselian local field k, the latter pairing defines an isomorphism between the groups $\mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{NS}\,\overline{X}, \overline{k}^*)$ and $\mathrm{Hom}_{\mathbf{Z}}((\mathrm{NS}\,\overline{X})^{\mathfrak{g}}, \mathbf{Q}/\mathbf{Z})$ (see [32, Chap. I.2, Ths. 2.1, 2.14]). Thus $e(X) = 0$. □

Remarks
(1) Let X be a smooth, projective, geometrically integral k-variety. Recall that the existence of a zero-cycle of degree 1 on X implies that $ob(X) = 0$ (see Lem. 2.1(ii)). If X is a curve over a p-adic field, the converse is also true by a theorem of Roquette and Lichtenbaum [26]. For X of arbitrary dimension over a p-adic field, and under the assumption that X has a regular model $\mathscr{X}$ proper over the ring of integers of k, it is possible to conjecture the equivalence of the two statements:
(a) there exists a zero-cycle of degree 1 on X;
(b) the map $\operatorname{Br} k \to \operatorname{Br} X/\operatorname{Br} \mathscr{X}$ is injective.
It is known (see [10, Th. 3.1]) that (a) implies (b) and that (b) implies the existence of a zero-cycle of degree a power of p. The proof of this last result given in [10] was conditional upon the conjectured absolute purity for the prime-to-p part of the Brauer group of $\mathscr{X}$; that property is now known, thanks to results of Gabber (see [18]).

(2) Over a p-adic field k, for any integer $n \leq 8$, there exist smooth cubic hypersurfaces $X \subset \mathbf{P}^n_k$ which have no rational point and, hence, by a theorem of Coray [14], have no zero-cycles of degree 1. If the dimension of the hypersurface is at least 3, Lemma 2.2(ix) gives $ob(X) = 0$.

(3) Theorem 2.5 as it stands does not extend to arbitrary fields k of cohomological dimension 2. Let $k = \mathbf{C}(u, v)$ be the rational function field in two variables. The quadric $Q \subset \mathbf{P}^3_k$ given by

$$X^2 + uY^2 + vZ^2 + (1+u)uvT^2 = 0$$

has no k-points, as one sees by going over to $\mathbf{C}((u))((v))$, but it satisfies $\operatorname{Br} k \hookrightarrow \operatorname{Br} k(Q)$. For $K = k(\sqrt{1+u})$, the group $\operatorname{Br} K$ does not inject into $\operatorname{Br} K(Q)$ (for more on this example, see Sec. 3.4).

Recall that a field R is real closed if -1 is not a sum of squares in R but is a sum of squares in any finite extension of R. By the Artin-Schreier theorem, $[\overline{R} : R] = 2$.

THEOREM 2.6
Let X be a geometrically integral variety over a real closed field R. Then $ob(X) = 0$ if and only if the natural map $\operatorname{Br} R \to \operatorname{Br} R(X)$ *is injective.*

Proof
The proof is the same as the proof of Theorem 2.5, once one takes into account the following two results.

Let A and B be dual abelian varieties over the field R. Let C be the algebraic closure of R. The natural pairing

$$A(R) \times \mathrm{H}^1(R, B) \to \operatorname{Br} R = \mathbf{Z}/2 \subset \mathbf{Q}/\mathbf{Z}$$

induces a perfect pairing of finite 2-torsion groups

$$A(R)/N_{C/R}A(C) \times \mathrm{H}^1(R, B) \to \mathbf{Q}/\mathbf{Z}$$

(over $R = \mathbf{R}$, see [32, Chap. I.3, Rem. 3.7]; in the general case, see [20]).

Let $\mathfrak{g} = \mathrm{Gal}\,(C/R)$. Let M be a finitely generated $\mathfrak{g}$-module. Then the natural pairing

$$M^{\mathfrak{g}} \times \mathrm{Ext}^2_{\mathfrak{g}}(M, C^*) \to \mathrm{Br}\, R = \mathbf{Z}/2$$

induces an isomorphism

$$\mathrm{Ext}^2_{\mathfrak{g}}(M, C^*) \simeq \mathrm{Hom}_{\mathbf{Z}}(M^{\mathfrak{g}}/N_{C/R}M, \mathbf{Z}/2)$$

(see [32, Chap. I.2, Th. 2.13]; the proof is given for $R = \mathbf{R}$, but it holds for an arbitrary real closed field). □

Remark. It is easy to give examples of varieties X over an arbitrary real closed field R such that $ob(X) = 0$ but $X(R) = \emptyset$ (e.g., anisotropic quadrics in $\mathbf{P}^n$ for $n \geq 4$). However, it is known that a smooth, geometrically integral R-variety X has an R-point if and only if, for all i, the maps $\mathrm{H}^i(R, \mathbf{Z}/2) \to \mathrm{H}^i(R(X), \mathbf{Z}/2)$ are injective. That the first statement implies the second is a general fact for smooth varieties over a field, with a rational point, which may be seen in a number of ways. If X/R is geometrically integral of dimension d and has no R-point, then the cohomological dimension of the field $R(X)$ is equal to d. This is a consequence of a theorem of Serre (see [9, Prop. 1.2.1]); for modern developments of this classical topic, see [37]).

We give a short, new proof of the following theorems of van Hamel (see [45, Sec. 5] for k the field of real numbers; see [46] for k a p-adic field). This theorem generalizes previous results of Roquette and of Lichtenbaum [26].

THEOREM 2.7 (van Hamel)
Let X be a smooth, projective, geometrically integral variety over a Henselian local field k or over a real closed field. Then $ob(X) = 0$ implies that $\delta(X) = 0$. In particular, a k-torsor X of an abelian variety is trivial if and only if $ob(X) = 0$.

Proof
Consider diagram (9). As recalled in Proposition 2.4, the image of $e(X) \in \mathrm{Ext}^2_{\mathfrak{g}}(\mathrm{Pic}\,\overline{X}, \overline{k}^*)$ in $\mathrm{Ext}^2_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*)$ is equal to the image of $\delta(X) \in \mathrm{H}^1(k, A)$ in $\mathrm{Ext}^2_{\mathfrak{g}}(J(\overline{k}), \overline{k}^*)$. The hypothesis $ob(X) = 0$ implies that $e(X) = 0$ (see Lem. 2.2(i)). Hence $J(k)$ is orthogonal to $\delta(X)$ with respect to the bottom pairing of (9). Since k is either a Henselian local field or a real closed field, Tate's second duality theorem implies that $\delta(X) = 0$. □

Let k be a Henselian local field, and let $\hat{k}$ be its completion. The following lemma is well known.

LEMMA 2.8
Let the fields k and $\hat{k}$ be as above. The natural map $\mathrm{Br}\, k \to \mathrm{Br}\, \hat{k}$ *is an isomorphism.*

The following result is due to Greenberg.

PROPOSITION 2.9 (see [21]])
Let the fields k and $\hat{k}$ be as above. If a k-algebra of finite type admits a k-algebra homomorphism to $\hat{k}$, then it admits a k-algebra homomorphism to k. In particular, the field $\hat{k}$ is the union of its k-subalgebras of finite type A admitting a retraction $A \to k$.

This implies that for any contravariant functor F from k-schemes to sets which commutes with filtering limits with affine transition morphisms, the natural map $F(X) \to F(X \times_k \hat{k})$ is injective. In particular, this applies to the functor $F(X) = \mathrm{Br}\, X$. This also implies that for any k-variety X, the conditions $X(k) \neq \emptyset$ and $X(\hat{k}) \neq \emptyset$ are equivalent.

PROPOSITION 2.10
Let the fields k and $\hat{k}$ be as above. Let X be a smooth, geometrically integral variety over k. Then $ob(X) = 0$ if and only if $ob(X \times_k \hat{k}) = 0$.

Proof
The previous comment implies that the map $\mathrm{Br}\, X \to \mathrm{Br}\,(X \times_k \hat{k})$ is injective. Together with Lemma 2.8, this shows that the map $\mathrm{Br}\, k \to \mathrm{Br}\, X$ is injective if and only if the map $\mathrm{Br}\, \hat{k} \to \mathrm{Br}\,(X \times_k \hat{k})$ is injective. In turn, this implies that $\mathrm{Br}\, k \to \mathrm{Br}\, k(X)$ is injective if and only if $\mathrm{Br}\, \hat{k} \to \mathrm{Br}\, \hat{k}(X)$ is injective. A double application of Theorem 2.5 completes the proof. □

Now, let $k \subset R$ be an inclusion of real closed fields. The analogue of Greenberg's result is a classical theorem going back to E. Artin: if a k-algebra of finite type admits a k-homomorphism to R, then it admits a k-homomorphism to k. The natural map $\mathrm{Br}\, k \to \mathrm{Br}\, R = \mathbf{Z}/2$ is a bijection. Theorem 2.6 and the same argument as above now give the following.

PROPOSITION 2.11
Let $k \subset R$ be an inclusion of real closed fields. Let X be a smooth, geometrically integral variety over k. Then $ob(X) = 0$ if and only if $ob(X \times_k R) = 0$.

2.3. The Brauer group and the elementary obstruction over number fields

PROPOSITION 2.12

Let X be a smooth, geometrically integral variety over a number field k, and let k_v be the completion of k at a place v. Then $ob(X) = 0$ implies that $ob(X \times_k k_v) = 0$.

Proof

Let $\tilde{k}$ be the integral closure of k in k_v. For v finite, this is the fraction field of the Henselization of the ring of integers of k at v. For v real, this is a real closed field. Since $\tilde{k} \subset \overline{k}$, the condition $ob(X) = 0$ implies that $ob(X \times_k \tilde{k}) = 0$. Now, the statement follows from Propositions 2.10 and 2.11. □

Recall that, by definition,

$$Б(X) = \mathrm{Ker}\Big[\mathrm{Br}_1 X \to \prod_v \mathrm{Br}_1 X_v / \mathrm{Br}_0 X_v\Big],$$

where $\mathrm{Br}_0 X_v$ is the image of $\mathrm{Br}\, k_v$ in $\mathrm{Br}_1 X_v$. This group does not change under restriction of X to a nonempty open set (see [36, Lem. 6.1]).

Recall that $X(\mathbb{A}_k)^{Б}$ is the subset of $X(\mathbb{A}_k)$ consisting of the adelic points orthogonal to $Б(X)$ with respect to the Brauer-Manin pairing (see [41, Sec. 5.2] for more details). Obviously, this set either is empty or coincides with $X(\mathbb{A}_k)$.

THEOREM 2.13

Let X be a smooth, geometrically integral variety over a number field k. Assume that $X(\mathbb{A}_k) \neq \emptyset$ and that $ob(X) = 0$. Then $X(\mathbb{A}_k) = X(\mathbb{A}_k)^{Б}$. In particular, $X(\mathbb{A}_k)^{Б} \neq \emptyset$.

Proof

Let us fix a Galois-equivariant section σ of the map $\overline{k}^* \to \overline{k}(X)^*$. For each place v of k, fix a decomposition group $\mathfrak{g}_v \subset \mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$. Let $\tilde{k}_v \subset \overline{k}$ be the fixed field of $\mathfrak{g}_v$. If v is finite, this is a Henselian local field. If v is a real place of k, then this is a real closure of k. Let $\alpha \in Б(X)$. For each place v of k, the image of α in $\mathrm{Br}\, X_v$ comes from a well-defined element of $\mathrm{Br}\, k_v$. Using the same arguments as at the end of Section 2.2, we see that the restriction of α to $\mathrm{Br}\,(X \times_k \tilde{k}_v)$ comes from a well-defined element ξ_v of $\mathrm{Br}\, \tilde{k}_v$. This last element may be computed by composing the maps

$$\mathrm{Br}_1(X \times_k \tilde{k}_v) \to \mathrm{H}^2\big(\mathfrak{g}_v, \overline{k}(X)^*\big) \to \mathrm{H}^2(\mathfrak{g}_v, \overline{k}^*),$$

where the last map is given by σ. We also have the element $\xi \in \mathrm{Br}\, k$, which is the image of α under the composite map $\mathrm{Br}_1 X \to \mathrm{H}^2(\mathfrak{g}, \overline{k}(X)^*) \to \mathrm{H}^2(\mathfrak{g}, \overline{k}^*)$, where the last map is induced by σ. Now, ξ_v is clearly the restriction of $\xi \in \mathrm{Br}\, k$ to $\mathrm{Br}\, \tilde{k}_v$. Thus the sum of the local invariants associated to the family ξ_v is the sum of the local invariants of ξ; it is therefore zero. □

Remark. We keep the assumption that $X(\mathbb{A}_k) \neq \emptyset$. In the particular case when $\operatorname{Pic} \overline{X}$ is a free abelian group, a delicate theorem asserts that the conditions $ob(X) = 0$ and $X(\mathbb{A}_k)^{Б} = X(\mathbb{A}_k)$ are equivalent (see [11, Prop. 3.3.2]). It would be interesting to see if the same is true in general.†

We conclude this section with the following observation, which does not seem to be documented in the literature (but see [29, Cor. 1, p. 40] for a similar result).

PROPOSITION 2.14
Let X be a smooth, proper, geometrically integral variety over a number field k, and let $A = \operatorname{Pic}^0 X$ be its Picard variety. Assume that for any finite extension K/k, the Tate-Shafarevich group of A_K is finite. Then the quotient of $Б(X)$ by the image of $\operatorname{Br} k$ *is finite.*

Proof
We have the exact sequence of Galois modules

$$0 \to \operatorname{Pic}^0 \overline{X} \to \operatorname{Pic} \overline{X} \to \operatorname{NS} \overline{X} \to 0.$$

Let K/k be a finite Galois extension such that $X(K) \neq \emptyset$ and the composite map

$$\operatorname{Pic} X_K \to \operatorname{Pic} \overline{X} \to \operatorname{NS} \overline{X}$$

is onto. Let $\mathfrak{h}$ be the Galois group of $\bar{k}$ over K. The $\mathfrak{h}$-module $\operatorname{NS} \overline{X}$ is the direct sum of a free abelian group $\mathbf{Z}^r$ and a finite abelian group F, both with trivial action of $\mathfrak{h}$. Galois cohomology yields the exact sequence

$$0 \to \mathrm{H}^1(K, \operatorname{Pic}^0 \overline{X}) \to \mathrm{H}^1(K, \operatorname{Pic} \overline{X}) \to \mathrm{H}^1(K, F).$$

We have analogous exact sequences over each Henselization $\tilde{K}_w$ of K:

$$0 \to \mathrm{H}^1(\tilde{K}_w, \operatorname{Pic}^0 \overline{X}) \to \mathrm{H}^1(\tilde{K}_w, \operatorname{Pic} \overline{X}) \to \mathrm{H}^1(\tilde{K}_w, F).$$

By Chebotarev's theorem, the kernel of the diagonal map $\mathrm{H}^1(K, F) \to \prod_w \mathrm{H}^1(\tilde{K}_w, F)$, where w runs through all places of K, vanishes. By our assumption on Tate-Shafarevich groups, the kernel of $\mathrm{H}^1(K, \operatorname{Pic}^0 \overline{X}) \to \prod_w \mathrm{H}^1(\tilde{K}_w, \operatorname{Pic}^0 \overline{X})$ is finite. Thus the kernel of $\mathrm{H}^1(K, \operatorname{Pic} \overline{X}) \to \prod_w \mathrm{H}^1(\tilde{K}_w, \operatorname{Pic} \overline{X})$ is finite.

†Wittenberg [49, Th. 3.3.1], building on work of Harari and Szamuely [25], has now proved that if one grants the finiteness of Tate-Shafarevich groups of abelian varieties over number fields, then the answer to this question is positive.

Let G be the finite group $\mathrm{Gal}(K/k)$. We have the standard restriction-inflation exact sequence

$$0 \to \mathrm{H}^1\big(G, (\mathrm{Pic}\,\overline{X})^{\mathfrak{h}}\big) \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) \to \mathrm{H}^1(K, \mathrm{Pic}\,\overline{X}).$$

The Mordell-Weil theorem and the Néron-Severi theorem imply that the abelian group $\mathrm{Pic}\,X_K = (\mathrm{Pic}\,\overline{X})^{\mathfrak{h}}$ is of finite type. Thus $\mathrm{H}^1(G, (\mathrm{Pic}\,\overline{X})^{\mathfrak{h}})$ is finite. It is then clear that the kernel of $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) \to \prod_v \mathrm{H}^1(\tilde{k}_v, \mathrm{Pic}\,\overline{X})$ is finite.

The argument given in the proof of Theorem 2.13 shows that the group $\text{Б}(X)$ may also be defined as

$$\text{Б}(X) = \mathrm{Ker}\Big[\mathrm{Br}_1 X \to \prod_v \mathrm{Br}_1 X_{\tilde{k}_v}/\mathrm{Br}_0 X_{\tilde{k}_v}\Big],$$

where $\mathrm{Br}_0 X_{\tilde{k}_v}$ is the image of $\mathrm{Br}\,\tilde{k}_v$ in $\mathrm{Br}_1 X_{\tilde{k}_v}$.

From the Hochschild-Serre spectral sequence for the multiplicative group and the projection map $X \to \mathrm{Spec}\,k$, we have the standard exact sequences

$$0 \to \mathrm{Br}_0 X \to \mathrm{Br}_1 X \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$$

and for each place v of k,

$$0 \to \mathrm{Br}_0 X_{\tilde{k}_v} \to \mathrm{Br}_1 X_{\tilde{k}_v} \to \mathrm{H}^1(\tilde{k}_v, \mathrm{Pic}\,\overline{X}).$$

The group $\text{Б}(X)/\mathrm{Br}_0 X$ is thus a subgroup of the kernel of the diagonal map $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) \to \prod_v \mathrm{H}^1(\tilde{k}_v, \mathrm{Pic}\,\overline{X})$. It is thus finite. □

3. Homogeneous spaces

By convention, all homogeneous spaces that we consider are right homogeneous spaces.

3.1. Structure of algebraic groups

Let k be a field of characteristic zero.

The following theorem is constantly used in this article. If $H \hookrightarrow G$ is a homomorphism of (not necessarily affine) algebraic groups over k which is an immersion, then the quotient G/H exists in the category of k-varieties (see Grothendieck [23, Th. 7.2, Cor. 7.4], Gabriel [19, Th. 3.2, p. 302]).

We also use the following fact: if $H \subset G$ is a normal subgroup of an algebraic group over k, and if X is a k-variety that is a right homogeneous space of G, then the quotient variety $Y = X/H$ exists in the category of k-varieties, it is a (right) G/H-homogeneous space, and the morphism $X \to Y$ is faithfully flat and smooth. When G is affine, a proof of this fact is given in [3, Lem. 3.1]. By the result of

Grothendieck and Gabriel mentioned above, that proof works for arbitrary algebraic groups.

If L is a connected linear group, we denote by L^{u} its unipotent radical, a normal, connected subgroup of L. We let L^{red} be the quotient of L by its unipotent radical L^{u}. This is a connected reductive group. We let $L^{\mathrm{ss}} \subset L^{\mathrm{red}}$ be the derived group of L^{red}. This is a connected semisimple group. We denote by L^{tor} the biggest toric quotient of L. The kernel of $L \to L^{\mathrm{tor}}$ is a normal, connected subgroup of L denoted by L^{ssu}. The group L^{ssu} is an extension of L^{ss} by L^{u}.

Any connected algebraic group G over k is an extension

$$1 \to L \to G \to A \to 1 \tag{12}$$

of an abelian variety A/k by a normal, connected linear k-group L (Chevalley's theorem; see [35], [13]). We write $L = G^{\mathrm{lin}}$. This is a characteristic subgroup of G; it is stable under all automorphisms of the group G. We denote by $Z(G)$ the centre of G and by G^{sab} the biggest group quotient of G, which is a semiabelian variety. We write G^{der} for the derived subgroup $[G, G]$. The group G^{der} is clearly contained in L, and thus it is a connected linear algebraic group.

If L is reductive, then $L^{\mathrm{der}} = G^{\mathrm{der}}$; in particular, G^{der} is a semisimple group. Indeed, the connected semisimple group L^{der} is normal in G; the quotient G' of G by L^{der} is an extension of A by the group L/L^{der}, which is L^{tor}. Any group extension of an abelian variety by a torus is central. Since there are no nonconstant morphisms from an abelian variety to a torus, any such group extension is commutative. Thus G' is a semiabelian variety. Since the kernel L^{der} of $G \to G'$ is semisimple, we have $G' = G^{\mathrm{sab}}$ and $L^{\mathrm{der}} = G^{\mathrm{der}}$.

By [50, Prop. 4], the connected group $G/Z(G)$ is linear. According to [50, Th. 1], we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}^1(k, J) \times & A(k) & \longrightarrow & \mathrm{Br}\, k \\ \downarrow & \downarrow & & \| \\ \mathrm{H}^1(k, J) \times & \mathrm{Ext}^1_{\mathfrak{g}}\big(J(\bar{k}), \bar{k}^*\big) & \longrightarrow & \mathrm{Br}\, k \end{array} \tag{13}$$

Let H be a linear k-group (not necessarily connected). We write $\hat{H}$ for the group of characters of $\overline{H}$ (this is a finitely generated discrete Galois module), and we write H^{mult} for the biggest quotient of H which is a k-group of multiplicative type. By construction, the k-groups H and H^{mult} have the same groups of characters. We set

$$H_1 = \ker[H \to H^{\mathrm{mult}}].$$

In Theorems 3.5, 3.11, and A.5, we make the hypothesis that H_1 is connected and that $\hat{H}_1 = 0$. This hypothesis is satisfied if H is connected. Indeed, in this case, the group H_1 coincides with the connected group H^{ssu}, and clearly $\overline{H}^{\mathrm{ssu}}$ has no nontrivial characters. For general H, the hypothesis need not be satisfied: consider the example where H is a finite, noncommutative, solvable group or the case of a noncommutative extension of a finite abelian group by a torus.

PROPOSITION 3.1

Let X be a homogeneous space of a connected k-group whose maximal connected linear subgroup has trivial unipotent radical. Assume that the stabilizers of the geometric points of X are connected. Then X can be given the structure of a homogeneous space of an algebraic group G satisfying the following conditions:

(i) G^{lin} *has trivial unipotent radical;*

(ii) G^{der} *is semisimple simply connected;*

(iii) *the stabilizers of the geometric points of X in G are linear and connected.*

Proof

Let G be a connected group whose maximal connected linear subgroup L has trivial unipotent radical. Assume that G acts transitively on X with connected geometric stabilizers. The group G is an extension (12). According to (13), we have $L/Z(L) = G/Z(G)$. Since L is reductive, the latter group is semisimple. This also implies that $G^{\mathrm{der}} = L^{\mathrm{der}}$, as explained above.

We write $\mathrm{St}_{\overline{x},\overline{G}}$ for the stabilizer of $\overline{x} \in X(\overline{k})$ in $\overline{G}$. These subgroups of $\overline{G}$ form one conjugacy class.

First reduction

The subgroup $Z(\overline{G}) \cap \mathrm{St}_{\overline{x},\overline{G}}$ is central in $\overline{G}$ and does not depend on $\overline{x}$. Hence it is stable under the action of the absolute Galois group $\mathfrak{g}$, and so $Z(\overline{G}) \cap \mathrm{St}_{\overline{x},\overline{G}} = \overline{C}$ for a central subgroup $C \subset G$. Then X is a homogeneous space of G/C such that $\mathrm{St}_{\overline{x},\overline{G/C}} = \mathrm{St}_{\overline{x},\overline{G}}/\overline{C}$. The group $G/Z(G)$ is linear; hence $\mathrm{St}_{\overline{x},\overline{G}}/\overline{C}$ is also linear. Replacing G by G/C, we may thus assume without loss of generality that the stabilizers of the geometric points are linear and connected.

Second reduction

It is well known (see [33, Prop. 3.1]) that given the connected reductive group $L/Z(L)$, there exist exact sequences of connected reductive algebraic groups

$$1 \to S \to H \to L/Z(L) \to 1$$

with S a k-torus central in H, and H^{der} simply connected. (Such extensions are called z-extensions.) Define G' as the fibred product of G and H over $L/Z(L)$, so that there is a commutative diagram of exact sequences of algebraic groups

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & S \arrow[r, phantom, "="] \arrow[d] & S \arrow[d] & \\
1 \arrow[r] & Z(G) \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(G) \arrow[r] & G \arrow[r] \arrow[d] & L/Z(L) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Note that $Z(G)$ is in the centre of G'. We then have the commutative diagram of exact sequences of connected linear algebraic groups

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & S \arrow[r, phantom, "="] \arrow[d] & S \arrow[d] & & \\
1 \arrow[r] & L' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & L \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & 1 & 1 & &
\end{tikzcd}$$

where L' is the kernel of the composite map $G' \to G \to A$. Clearly, L' is linear, so it is the maximal linear subgroup of G'. Thus the natural map $L'/Z(L') \to G'/Z(G')$ is an isomorphism of semisimple groups. Since $Z(G)$ is a central subgroup of G', the map $G' \to G'/Z(G')$ factors as $G' \to H \to G'/Z(G')$. The maps $L' \to G' \to H \to G'/Z(G')$ give rise to a series of maps

$$(L')^{\mathrm{der}} \to (G')^{\mathrm{der}} \to H^{\mathrm{der}} \to \big(G'/Z(G')\big)^{\mathrm{der}} = L'/Z(L'), \tag{14}$$

where the composite map is induced by the natural map $L' \to L'/Z(L')$. Since L' is reductive, the first map in (14) is an isomorphism, as explained above. The maps $G' \to H \to G'/Z(G')$ are surjective; hence so are the second and the third maps in (14). Since L' is a reductive group, the natural map $(L')^{\mathrm{der}} \to L'/Z(L')$ is an isogeny;

hence $(L')^{\mathrm{der}} \to H^{\mathrm{der}}$ is also an isogeny. But H^{der} is simply connected since H is a z-extension. This forces $(L')^{\mathrm{der}} \simeq H^{\mathrm{der}}$, so that $(L')^{\mathrm{der}} = (G')^{\mathrm{der}}$ is a semisimple simply connected group. Replacing G by G', we keep the property that the stabilizers of the geometric points are connected linear groups. □

3.2. Local fields: Semiabelian varieties

THEOREM 3.2

Let k be a Henselian local field or a real closed field. A k-torsor X of a semiabelian variety is trivial if and only if $ob(X) = 0$.

Proof

Let X be a torsor of a semiabelian variety G, an extension of an abelian variety A by a torus T:

$$1 \to T \to G \to A \to 0.$$

Let D be the quotient of X by the action of T; this is a k-torsor of A, which can also be defined as the pushforward of X with respect to the map $G \to A$. By functoriality, $ob(D) = 0$, so that $D \simeq A$ by Theorem 2.7. Thus X is an A-torsor of T. We write ξ for the class of this torsor in $\mathrm{H}^1_{\text{ét}}(A, T)$, and we write $\xi_m \in \mathrm{H}^1(k, T)$ for the class of the fibre X_m at a k-point m of A. Our goal is to find m with $\xi_m = 0$.

From the bilinear pairing of k-group schemes

$$\hat{T} \times T \to \mathbf{G}_{m,k},$$

we deduce a cup-product pairing

$$\mathrm{H}^1(k, \hat{T}) \times \mathrm{H}^1_{\text{ét}}(A, T) \to \mathrm{H}^2_{\text{ét}}(A, \mathbf{G}_m) = \mathrm{Br}\, A,$$

the image of which lies in $\mathrm{Br}_1 A$.

Let $B \subset \mathrm{Br}_1 A$ be the subgroup consisting of the elements $\alpha \cup \xi$, where $\alpha \in \mathrm{H}^1(k, \hat{T})$. The group $\mathrm{H}^1(k, \hat{T})$ is finite; hence so is B.

The k-point $0 \in A(k)$ defines a splitting of (2) applied to $X = A$, so that $\mathrm{Br}_1 A$ decomposes as the direct sum of $\mathrm{Br}\, k$ and the subgroup consisting of the elements $\mathscr{A} \in \mathrm{Br}_1 A$ such that $\mathscr{A}(0) = 0$, naturally identified with $\mathrm{H}^1(k, \mathrm{Pic}\, \overline{A})$. The canonical map $r : \mathrm{Br}_1 A \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{A})$ can be written as $\mathscr{A} \mapsto \mathscr{A} - \mathscr{A}(0)$. Let J be the Picard variety of A, which is also the dual abelian variety of A.

We now prove the following statements, the last of which proves the theorem.

(1) The restriction of the canonical map $r : \mathrm{Br}_1 A \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{A})$ to B factors through $\mathrm{H}^1(k, J)$.

(2) We have $B \cap \mathrm{Br}\, k = 0$.

(3) There exists a point $m \in A(k)$ orthogonal to the group B with respect to the pairing $A(k) \times \mathrm{Br}\, A \to \mathrm{Br}\, k$ given by the evaluation.

(4) For any point m satisfying (3), we have $\xi_m = 0$; that is, the fibre of $X \to A$ over m contains a k-point.

Proof of (1). Let $\lambda : \hat{T} \to \mathrm{Pic}\, \overline{A}$ be the type of the torsor $X \to A$ (see [11, (2.0.2)], [41]). It is well known (see [38, Chap. VII, no. 16, Th. 6 and the comment thereafter]) that $\overline{X}$ can be given the structure of a group extension of $\overline{A}$ by $\overline{T}$ if and only if λ factors through the natural injection $J(\overline{k}) \hookrightarrow \mathrm{Pic}\, \overline{A}$. Now, (1) follows from [41, Th. 4.1.1], which says that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^1(k, \hat{T}) & \xrightarrow{\lambda_*} & \mathrm{H}^1(k, J) \\ \big\downarrow{\scriptstyle \cup\xi} & & \big\downarrow \\ \mathrm{Br}_1 A & \xrightarrow{r} & \mathrm{H}^1(k, \mathrm{Pic}\, \overline{A}) \end{array} \tag{15}$$

Proof of (2). The image of ξ under the base change map $\mathrm{H}^1(A, T) \to \mathrm{H}^1(X, T)$ is zero since $X \times_A X$ is a trivial X-torsor (the diagonal is a section). Thus B goes to zero under the pullback map $\mathrm{Br}\, A \to \mathrm{Br}\, X$. The assumption $ob(X) = 0$ implies that the natural map $\mathrm{Br}\, k \to \mathrm{Br}\, X$ is injective, and this implies that $B \cap \mathrm{Br}\, k = 0$.

Proof of (3). We now *define* a pairing

$$A(k) \times \mathrm{H}^1(k, \hat{T}) \to \mathrm{Br}\, k \tag{16}$$

in the following manner. A couple $(m, \alpha) \in A(k) \times \mathrm{H}^1(k, \hat{T})$ is sent to

$$(\alpha \cup \xi)(m) - (\alpha \cup \xi)(0) = \alpha \cup (\xi_m - \xi_0).$$

We claim that this pairing is bilinear. To prove this, consider the diagram of pairings

$$\begin{array}{ccccc} A(k) & \times & \mathrm{H}^1(k, J) & \longrightarrow & \mathrm{Br}\, k \\ \| & & \big\downarrow & & \| \\ A(k) & \times & \mathrm{H}^1(k, \mathrm{Pic}\, \overline{A}) & \longrightarrow & \mathrm{Br}\, k \end{array} \tag{17}$$

where the top row is the Tate pairing and the bottom row is the pairing given by evaluating elements of $\mathrm{H}^1(k, \mathrm{Pic}\, \overline{A})$, understood as the subgroup of $\mathrm{Br}_1 A$ consisting of the elements with trivial value at zero. This diagram commutes by [28, Prop. 8(c)]. From (15), we see that the map $\mathrm{H}^1(k, \hat{T}) \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{A})$ sending α to $r(\alpha \cup \xi) = \alpha \cup \xi - (\alpha \cup \xi)(0)$ factors through $\mathrm{H}^1(k, J)$, and so the pairing (16) is bilinear since such is the top pairing of (17).

We now use the hypothesis on the field k. There is a natural embedding $\operatorname{Br} k \hookrightarrow \mathbf{Q}/\mathbf{Z}$. The pairing (16) induces a homomorphism $\sigma : A(k) \to B^* = \operatorname{Hom}(B, \mathbf{Q}/\mathbf{Z})$. Let us show that σ is surjective. If it is not, there exists $b \in B$, $b \neq 0$, such that $\sigma(m)$ applied to $b = \xi \cup \alpha \in \operatorname{Br}_1 A$ is zero for any m; that is, $b(m) - b(0) = 0$ for all $m \in A(k)$. Thus $b - b(0)$ comes from an element of $\mathrm{H}^1(k, J)$ orthogonal to $A(k)$ with respect to the Tate pairing. However, over a Henselian local field or a real closed field, the right kernel of the Tate pairing $A(k) \times \mathrm{H}^1(k, J) \to \operatorname{Br} k$ is zero; hence $b = b(0) \in B \subset \operatorname{Br}_1 A$ is a nonzero constant element in B. This contradicts (2).

By the surjectivity of σ, there exists $m \in A(k)$ such that $\sigma(m)$ is the element of B^* given by $b \mapsto -b(0)$ for any $b \in B$. This says that $b(m) - b(0) = -b(0)$, so that $b(m) = 0$ for any $b \in B$. This finishes the proof of (3).

Proof of (4). By (3) we have $(\alpha \cup \xi)(m) = \alpha \cup \xi_m = 0$ for any $\alpha \in \mathrm{H}^1(k, \hat{T})$. Hence ξ_m is orthogonal to $\mathrm{H}^1(k, \hat{T})$ with respect to the pairing

$$\mathrm{H}^1(k, \hat{T}) \times \mathrm{H}^1(k, T) \to \operatorname{Br} k.$$

For k a Henselian local field or a real closed field, this pairing is nondegenerate (see [32, Chap. I.2, Ths. 2.14(c), 2.13], in which the proof of Th. 2.13 works over a real closed field); thus $\xi_m = 0$. This finishes the proof of the theorem. □

3.3. p-adic fields: Main theorem

THEOREM 3.3
Let X/k be a homogeneous space of a connected k-group (not necessarily linear) such that the stabilizer $\overline{H}$ of a geometric point $\overline{x} \in X(\overline{k})$ is connected. If k is a Henselian local field, then X has a k-point if and only if $ob(X) = 0$.

In conjunction with Theorem 2.5, this gives the following corollary.

COROLLARY 3.4
Let X/k be a homogeneous space of a connected k-group (not necessarily linear) such that the stabilizer $\overline{H}$ of a geometric point $\overline{x} \in X(\overline{k})$ is connected. If k is a Henselian local field, then X has a k-point if and only if $\operatorname{Br} k$ *injects into* $\operatorname{Br} k(X)$.

Proof of Theorem 3.3
First reduction
Suppose that X is a right homogeneous space of a connected group G represented as an extension (12). The unipotent radical $L^{\mathrm{u}} \subset L$ is a normal subgroup of G. Let $G' = G/L^{\mathrm{u}}$. This group satisfies $(L')^{\mathrm{u}} = 0$. The following properties, proved in [3, Lem. 3.1], hold over any perfect field k. The quotient $X' = X/L^{\mathrm{u}}$ exists, and there is a natural projection map $X \to X'$. This map is surjective on $\overline{k}$-points, and its geometric

fibres are orbits of L^{u}. The variety X' is a homogeneous space of G' with connected geometric stabilizers.

The hypothesis $ob(X) = 0$ implies that $ob(X') = 0$. Suppose that we have found a k-point $y \in X'(k)$. Then the fibre X_y is a k-variety that is a homogeneous space of the unipotent k-group L^{u}. According to [3, Lem. 3.2(i)], over any perfect field k this implies that $X_y(k) \neq \emptyset$. Thus $X(k) \neq \emptyset$.

Thus without loss of generality, we may assume that the unipotent radical L^{u} of L is trivial, so that L is reductive.

Second reduction

By Proposition 3.1, we can further assume that G^{der} is semisimple simply connected and that the stabilizers of the geometric points of X in G are linear and connected. This reduction has nothing to do with the nature of the field k. It does not change X; hence we keep the assumption that $ob(X) = 0$.

Relaxing the assumptions

To prove Theorem 3.3, it is enough to prove the following result (whose proof is similar to that of [3, Th. 2.2]). We write G^{ss} for L^{ss}, and we write G^{u} for L^{u}, where $L = G^{\mathrm{lin}}$. (The notation $\overline{H}_1$ was defined in Sec. 3.1.)

THEOREM 3.5

Let k be a Henselian local field, let G be a connected k-group, and let X/k be a homogeneous space of G with geometric stabilizer $\overline{H}$. Assume that

(i) $G^{\mathrm{u}} = \{1\}$*;*

(ii) $\overline{H} \subset \overline{G}^{\mathrm{lin}}$*;*

(iii) G^{ss} *is simply connected;*

(iv) $\overline{H}_1$ *is connected and has no nontrivial characters (e.g., $\overline{H}$ is connected).*

Then $ob(X) = 0$ if and only if $X(k) \neq \emptyset$.

The homogeneous space X defines a k-form of $\overline{H}^{\mathrm{mult}}$ which we denote by M (see [3, Sec. 4.1]). We have a canonical homomorphism $M \to G^{\mathrm{sab}}$ (for this, see the computation at the end of [4, Sec. 1.2]). In [4], $G = L$ is linear, and the calculation uses the commutativity of L^{tor}. It generalizes to the present context with the commutative group G^{sab} in place of L^{tor}.

Here is another way to construct the homomorphism $M \to G^{\mathrm{sab}}$. One extends the base field from k to the function field $k(X)$ of X. Consideration of the stabilizer H' of the generic point of X yields a map $H' \to G \times_k k(X)$ over $k(X)$ which induces a map $H' \to G^{\mathrm{sab}} \times_k k(X)$. Since H (and hence H') is linear, this map factors through $T \times_k k(X)$, where $T = (G^{\mathrm{sab}})^{\mathrm{lin}}$ is the maximal torus inside the semiabelian variety

G^{sab}. There is then an induced $k(X)$-morphism $M \times_k k(X) \to T \times_k k(X)$. Such a map comes from a unique morphism $M \to T \subset G^{\mathrm{sab}}$.

We first prove a special case of Theorem 3.5.

PROPOSITION 3.6

With the hypotheses of Theorem 3.5, assume that M injects *into G^{sab} (in other words, $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$). Then X has a k-point.*

Proof

Set $Y = X/G^{\mathrm{ss}}$. Then Y is a homogeneous space of the semiabelian variety G^{sab}; hence it is a torsor of some semiabelian variety. We have a canonical map $X \to Y$. From $ob(X) = 0$, we deduce that $ob(Y) = 0$ (see the beginning of the introduction). By Theorem 3.2, Y has a k-point y. Let X_y denote the fibre of X over y. It is a homogeneous space of G^{ss} with geometric stabilizer $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$. The group G^{ss} is semisimple simply connected by (iii). The group $\overline{H}_1$ is connected and has no nontrivial characters by (iv). By [2, Th. 7.2] (that theorem is stated over a p-adic field, but it also holds over a Henselian local field; see the proof of Th. 3.8 hereafter), the k-variety X_y has a k-point. Hence X has a k-point. □

For the general case, we need an easy lemma.

LEMMA 3.7

Let M be a k-group of multiplicative type, and let $\eta \in \mathrm{H}^2(k, M)$ be a cohomology class. Then there exists an embedding $j : M \hookrightarrow P$ into a quasi-trivial k-torus P such that $j_*(\eta) = 0$.

Proof

We can embed M into a quasi-trivial torus, and so we assume without loss of generality that $M = R_{K/k}\mathbf{G}_m$ for some finite extension K/k. We have a canonical isomorphism $s_K : \mathrm{H}^2(k, R_{K/k}\mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^2(K, \mathbf{G}_m)$. Let L/K be a finite extension such that the image of $s_K(\eta)$ in $\mathrm{H}^2(L, \mathbf{G}_m)$ is zero. Consider the natural injection of quasi-trivial tori $c_{K/L} : R_{K/k}\mathbf{G}_m \hookrightarrow R_{L/k}\mathbf{G}_m$. Then $(c_{K/L})_*(\eta) = 0$. □

Let us resume the proof of Theorem 3.5. Let $\overline{x} \in X(\overline{k})$ be a point with stabilizer $\overline{H}$. Let $\eta_X \in \mathrm{H}^2(k, \overline{H}, \kappa)$ be the cohomology class (Springer's class) defined by X (see [2, Sec. 7.7] or [43, Sec. 1.20]), where κ is the k-kernel defined by X (see [2, Sec. 7.1]). Recall that $\overline{H}_1 = \ker[\overline{H} \to \overline{H}^{\mathrm{mult}}]$. Clearly, the subgroup $\overline{H}_1$ is invariant under all semialgebraic automorphisms of $\overline{H}$; hence κ induces a k-kernel κ^{mult} in $\overline{H}^{\mathrm{mult}}$, and we obtain a map

$$\mu_*\colon \mathrm{H}^2(k, \overline{H}, \kappa) \to \mathrm{H}^2(k, \overline{H}^{\mathrm{mult}}, \kappa^{\mathrm{mult}})$$

induced by the canonical map $\mu\colon \overline{H} \to \overline{H}^{\mathrm{mult}}$ (see [2, Sec. 1.7]). Since $\overline{H}^{\mathrm{mult}}$ is an abelian group, κ^{mult} defines a k-form of $\overline{H}^{\mathrm{mult}}$, which is the k-form M mentioned above. We obtain an element $\mu_*(\eta_X) \in \mathrm{H}^2(k, M) = \mathrm{H}^2(k, \overline{H}^{\mathrm{mult}}, \kappa^{\mathrm{mult}})$. Note that in [2, Sec. 7], G is assumed to be semisimple and simply connected, but the general constructions we refer to hold for an arbitrary k-group G; the key point is that the subgroup $\overline{H}$ is linear.

By Lemma 3.7, we can construct an embedding $j\colon M \hookrightarrow P$ into a quasi-trivial k-torus P such that $j_*(\mu_*(\eta_X)) = 0$. Consider the k-group $F = G \times P$, and consider the embedding

$$\overline{H} \hookrightarrow \overline{F} = F \times_k \overline{k} \quad \text{given by } h \mapsto \big(h, j(\mu(h))\big).$$

Set $\overline{Z} = \overline{H}\backslash\overline{F}$. We have a right action $\overline{a}\colon \overline{Z} \times \overline{F} \to \overline{Z}$ and an $\overline{F}$-equivariant map

$$\overline{\pi}\colon \overline{Z} \to \overline{X}, \qquad \overline{H} \cdot (g, p) \mapsto \overline{H} \cdot g, \quad \text{where } g \in \overline{G},\ p \in \overline{P}.$$

Then $\overline{Z}$ is a homogeneous space of $\overline{F}$ with respect to the action $\overline{a}$, and the map $\overline{\pi}\colon \overline{Z} \to \overline{X}$ is a torsor under $\overline{P}$. The homomorphism $M \to F^{\mathrm{sab}}$ is injective.

In [3, Sec. 4.7], it is proved that $\mathrm{Aut}_{\overline{F},\overline{X}}(\overline{Z}) = P(\overline{k})$. By [3, Lem. 4.8], the element $j_*(\mu_*(\eta_X)) \in \mathrm{H}^2(k, P)$ is the only obstruction to the existence of a k-form (Z, a, π) of the triple $(\overline{Z}, \overline{a}, \overline{\pi})$: there exists such a k-form if and only if $j_*(\mu_*(\eta_X)) = 0$. In our case, by construction we have $j_*(\mu_*(\eta_X)) = 0$; hence there exists a k-form (Z, a, π) of $(\overline{Z}, \overline{a}, \overline{\pi})$. Since $\pi : Z \to X$ is a torsor under the quasi-trivial torus P, from Hilbert's theorem 90 and Shapiro's lemma, we conclude that Z is k-birationally isomorphic to $X \times P$. From $ob(X) = 0$, we deduce $ob(Z) = 0$ (see the beginning of the introduction).

We obtain a homogeneous space Z of a connected k-group F such that F^{ss} is simply connected, with geometric stabilizer $\overline{H}$. The group M injects into the group $F^{\mathrm{sab}} = G^{\mathrm{sab}} \times P$, and $ob(Z) = 0$. By Proposition 3.6, Z has a k-point. Thus X has a k-point. $\square$

Remark. In [17, Th. 3.9], Florence constructs a homogeneous space X of the group $G = \mathrm{PGL}(D)$ for a quaternion algebra D over a p-adic field such that the geometric stabilizer $\overline{H} \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ and X has a zero-cycle of degree 1 but no rational points. The space X can also be viewed as a homogeneous space of $\mathrm{SL}(D)$, the geometric stabilizer now being the quaternion group. Since X has a zero-cycle of degree 1, the map $\mathrm{Br}\,k \to \mathrm{Br}\,k(X)$ is injective. Thus $ob(X) = 0$. This shows that in Theorem 3.5, neither condition (iii) nor condition (iv) may be omitted.

3.4. Good fields of cohomological dimension at most 2

A field of characteristic zero is called a good field of cohomological dimension at most 2 if it satisfies the following properties.

(i) Its cohomological dimension $\mathrm{cd}(k)$ is at most 2.

(ii) For any central simple algebra A over a finite field extension K of the field k, the index of A (as a K-algebra) and the exponent of the class of A in $\mathrm{Br}\,K$ coincide.

(iii) For any semisimple simply connected group G/k, we have $\mathrm{H}^1(k, G) = 0$.

According to Serre's conjecture II, (i) should imply (iii). This is known for groups of classical type. The combination of (i) and (ii) implies (iii) for all groups without factors of type E_8 (see the references in [6]).

Properties (i) to (iii) are satisfied for Henselian local fields and for totally imaginary number fields.

For the fraction field of a 2-dimensional strictly Henselian local domain with residue field of characteristic zero, these three properties also hold (see [8], [6]).

For the function field of an algebraic surface over an algebraically closed field of characteristic zero, properties (i) and (ii) hold. For (ii), this is de Jong's theorem [15]. Hence, in this case, (iii) is known when G has no factors of type E_8.

THEOREM 3.8

Let k be a good field of cohomological dimension at most 2 and characteristic zero. Let X/k be a homogeneous space of a connected linear *group G. Assume that the geometric stabilizers are connected. Then $X(k) \neq \emptyset$ if and only if $ob(X) = 0$.*

Proof

We follow the proof of Theorem 3.3. The first and second reduction have nothing to do with the nature of the field k. It remains to prove the analogue of Theorem 3.5. Since G here is linear, the semiabelian variety G^{sab} is a k-torus. With notation as in the proof of Proposition 3.6, the k-variety Y is a homogeneous space of a k-torus. It satisfies $ob(Y) = 0$. Over any field, this implies that $Y(k) \neq \emptyset$ (see Lem. 2.1(iv)). Keeping the notation of Proposition 3.6, one finds a point $y \in Y(k)$, and then the k-variety X_y is a homogeneous space of G^{ss} with geometric stabilizer $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$. The group G^{ss} is semisimple simply connected. The group $\overline{H}_1$ is connected and has no nontrivial characters. Over a good field of cohomological dimension 2, the analogue of [2, Th. 7.2] is [6, Props. 5.3, 5.4], which build upon the key theorem [6, Th. 2.1] and use the formalism of [2]. This shows that the k-variety X_y has a k-point. Hence X has a k-point. This completes the proof of the analogue of Proposition 3.6.

Lemma 3.7 holds over any field. The rest of the proof of Theorem 3.5 is a reduction to Proposition 3.6, which works equally well over any ground field. □

COROLLARY 3.9
Let k be a good field of cohomological dimension at most 2 and characteristic zero. Let X/k be a homogeneous space of a connected linear group G. Assume that the geometric stabilizers are connected.

(i) *Then $X(k) \neq \emptyset$ if and only if, for any flasque k-torus S, the restriction map $\mathrm{H}^2(k, S) \to \mathrm{H}^2(k(X), S)$ is injective.*
(ii) *If X is projective, then $X(k) \neq \emptyset$ if and only if, for any finite field extension K/k, the map $\mathrm{Br}\, K \to \mathrm{Br}\, K(X)$ is an injection.*
(iii) *If X is projective and the abelian group $\mathrm{Pic}\,(\overline{X})$ is free of rank 1, then $X(k) \neq \emptyset$ if and only if the natural map $\mathrm{Br}\, k \to \mathrm{Br}\, k(X)$ is an injection.*

Proof
(i) This follows from [7, th. A] and from Theorem 3.8 and Lemma 2.2(iv).
(ii) The Bruhat decomposition implies that the geometric Picard group of a projective homogeneous space of a connected linear group is a permutation $\mathfrak{g}$-module (cf. [6, proof of Lem. 5.6, p. 337]). Now (ii) follows from Theorem 3.8 and Lemma 2.2(vi).
(iii) This follows from Theorem 3.8 and Lemma 2.2(v). □

Remark (3) after Theorem 2.5 shows that in (ii) above, one cannot simply assume the injectivity of $\mathrm{Br}\, k \to \mathrm{Br}\, k(X)$.

Remarks
(1) For any even integer $n = 2m \geq 6$, Merkurjev [30] constructs a (big) field k_n of cohomological dimension 2 and an anisotropic quadratic form of rank n over k_n. The associated quadric is a homogeneous space of a spinor group with connected geometric stabilizers. There are elements of order 2 in the Brauer group of k_n which are not of index 2. Thus the mere hypothesis $\mathrm{cd}(k) \leq 2$ is not enough for the above theorem to hold; condition (ii) (in the definition of a good field of cohomological dimension at most 2) is required.
(2) The above corollary should be compared with the recent work of de Jong, He, and Starr [16] on projective homogeneous varieties over function fields in two variables.
(3) Let $k = \mathbf{C}(u, v)$ be the rational function field in two variables over the complex field. Let $X \subset \mathbf{P}^8_k$ be the smooth cubic hypersurface defined by the diagonal cubic form with coefficients $1, u, u^2, v, vu, vu^2, v^2, v^2u, v^2u^2$. One easily checks that $X(k) = \emptyset$. In fact, X has no points in $\mathbf{C}((u))((v))$. On

the other hand, Lemma 2.2(ix) ensures that $ob(X) = 0$. The same comment applies to smooth cubic hypersurfaces in $\mathbf{P}^n_k$ with $4 \leq n \leq 7$ defined by taking subforms of the above form.

3.5. Number fields

Let k be a number field. We write Ω_r for the set of all *real* places of k. We set $k_r = \prod_{v\in\Omega_r} k_v$. Then, for a k-variety X, we have $X(k_r) = \prod_{v\in\Omega_r} X(k_v)$. When k is totally imaginary, the following result is a special case of Theorem 3.8.

THEOREM 3.10

Let k be a number field, and let X/k be a homogeneous space of a connected linear *algebraic k-group G with connected geometric stabilizer. Assume that X has a k_v-point for every real place v of k. If $ob(X) = 0$, then X has a k-point.*

Proceeding as in Section 3.3, we see that this is a consequence of the following result, the proof of which is similar to that of [3, Th. 2.2].

THEOREM 3.11

Let k be a number field, and let X/k be a homogeneous space of a connected linear *algebraic k-group G with geometric stabilizer $\overline{H}$. Assume that*

(i) $G^{\mathrm{u}} = \{1\}$;

(ii) G^{ss} *is simply connected;*

(iii) $\overline{H}_1$ *is connected and has no nontrivial characters;*

(iv) *X has a k_v-point for every $v \in \Omega_r$.*

If $ob(X) = 0$, then X has a k-point.

The homogeneous space X defines a k-form of $\overline{H}^{\mathrm{mult}}$, which we denote by M. We have a canonical homomorphism $M \to G^{\mathrm{tor}}$. We first prove a special case of Theorem 3.11.

PROPOSITION 3.12

In Theorem 3.11, assume that M injects into G^{tor} (i.e., $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$). Then X has a k-point.

Proof

Set $Y = X/G^{\mathrm{ss}}$. Then Y is a homogeneous space of the k-torus G^{tor}; hence it is a torsor of some k-torus T. We have a canonical map $\alpha\colon X \to Y$. Since $ob(X) = 0$, we see that $ob(Y) = 0$. Hence Y has a k-point y by Lemma 2.1(iv).

The map $\alpha\colon X \to Y$ is smooth; hence, for $v \in \Omega_r$, the image $\mathscr{Y}_v := \alpha(X(k_v))$ is open in $Y(k_v)$ and nonempty (because X has a k_v-point). Set $\mathscr{Y}_r = \prod_{v\in\Omega_r} \mathscr{Y}_v$; then $\mathscr{Y}_r$ is a nonempty open subset in $Y(k_r)$. By the real approximation theorem for tori, which is due to Serre (see [36, cor. 3.5], [47, Th. 11.5]), the set $Y(k)$ is dense in $Y(k_r)$. Hence there exists a k-point $y' \in Y(k) \cap \mathcal{Y}_r$.

Consider the fibre $X_{y'}$ of X over y'. It is a homogeneous space of G^{ss} with geometric stabilizer $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$. The group G^{ss} is semisimple simply connected by assumption (ii) of Theorem 3.11. The group $\overline{H}_1$ is connected and has no nontrivial characters by assumption (iii) of Theorem 3.11. Since $y' \in \mathcal{Y}_r$, the variety $X_{y'}$ has a k_v-point for every $v \in \Omega_r$. By [2, Th. 7.3(vi), Cor. 7.4], $X_{y'}$ has a k-point. Hence X has a k-point. □

We resume the proof of Theorem 3.11. Let G and X be as in that theorem. Let $\overline{x} \in X(\overline{k})$ be a point with stabilizer $\overline{H}$. We have a canonical map $\mu_* \colon \mathrm{H}^2(k, \overline{H}, \kappa) \to \mathrm{H}^2(k, M)$, where κ is the k-kernel defined by X. Let $\eta_X \in \mathrm{H}^2(k, \overline{H}, \kappa)$ be the cohomology class defined by X. Consider $\mu_*(\eta_X) \in \mathrm{H}^2(k, M)$. By Lemma 3.7, we can construct an embedding $j \colon M \hookrightarrow P$ into a quasi-trivial k-torus P such that $j_*(\mu_*(\eta_X)) = 0$.

As in the proof of Theorem 3.5, we construct the k-group $F = G \times P$ and a triple (Z, a, π), where (Z, a) is a homogeneous space of F and (Z, π) is a torsor of P over X. Since (Z, π) is a torsor of the quasi-trivial torus P over X, and since X has a k_v-point for any $v \in \Omega_r$, we see that Z has a k_v-point for such v. Also, since (Z, π) is a torsor of the quasi-trivial torus P, we see that Z is k-birationally isomorphic to $X \times P$. Since $ob(X) = 0$, we see that $ob(Z) = 0$.

We obtain a homogeneous space Z of a connected reductive k-group F such that F^{ss} is simply connected with geometric stabilizer $\overline{H}$. The group M injects into $F^{\mathrm{tor}} = G^{\mathrm{tor}} \times P$, and $ob(Z) = 0$. The homogeneous space Z has a k_v-point for any $v \in \Omega_r$. By Proposition 3.12, Z has a k-point. Thus X has a k-point. □

Remark. To prove Theorem 3.10, one could also argue as follows. According to Proposition 2.12, the hypothesis $ob(X) = 0$ implies that $ob(X \times_k k_v) = 0$ for each nonarchimedean place v of k. Theorem 3.3 then implies that $X(k_v) \neq \emptyset$ for each nonarchimedean place v. Thus $X(\mathbb{A}_k) \neq \emptyset$. Theorem 2.13 then implies that $X^c(\mathbb{A}_k)^{\text{Б}} = X^c(\mathbb{A}_k) \neq \emptyset$. From [3, Th. 2.2], we conclude that $X(k) \neq \emptyset$. This proof looks more elegant than the one above, but it relies on [3, Th. 2.2], whose proof occupies most of that article. In the proof given above, one sees precisely where the linearity of G is used: it is to ensure weak approximation at the real places for Y, which is a principal homogeneous space of a torus (a similar argument occurs in [3]). Had we not assumed G to be linear, Y would have been a principal homogeneous space of a semiabelian variety. For an abelian variety A over a number field k, weak approximation at the real places may fail badly: over some real completion k_v, there may be no k-point in a connected component of $A(k_v) = A(\mathbf{R})$. This is the basis of the example given in Section 3.6.

The question as to whether the Brauer-Manin obstruction attached to $\text{Б}(X)$ is the only obstruction to the Hasse principle on k-torsors of arbitrary connected algebraic groups was raised in [41, Ques. 1, p. 133]. Harari and Szamuely recently gave a positive solution to this problem for torsors of semiabelian varieties.

THEOREM 3.13 (see Harari-Szamuely [25])
Let k be a number field, and let X be a k-torsor of a semiabelian variety G. Assume that the Tate-Shafarevich group of the biggest quotient of G which is an abelian variety is finite. If X has a family of local points $P_v \in X(k_v)$ (v running through the places of k) which is orthogonal to $Б(X)$ with respect to the Brauer-Manin pairing, then X has a k-point.

This implies the following global analogue of Theorem 3.3.

THEOREM 3.14
Let X be a homogeneous space of a (not necessarily linear) connected group G such that the stabilizers of the geometric points of X are connected. Assume that the Tate-Shafarevich group of the biggest quotient of G which is an abelian variety is finite. If k is a totally imaginary number field, then X has a k-point if and only if $ob(X) = 0$.

Proof
We follow the proof of Theorem 3.3 up to the place where Theorem 3.2 is used, and we apply Theorems 2.13 and 3.13 instead. Then [2, Th. 7.2] (local) and [2, Cor. 7.4] (global) allow us to finish the proof in the same way as before. □

3.6. Number fields: An example
We now proceed to construct a $\mathbf{Q}$-torsor X of a noncommutative connected algebraic group over $\mathbf{Q}$ such that $ob(X) = 0$, X has points over all completions of $\mathbf{Q}$, and further, $X^c(\mathbb{A}_{\mathbf{Q}})^{Б} = X^c(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$, but X has no $\mathbf{Q}$-points. Thus, in general, the answer to the aforementioned question is negative.

Let $E/\mathbf{Q}$ be the elliptic curve with affine equation

$$y^2 = (x^2 - 3)(x - 2).$$

We note that the set $E(\mathbf{R})$ has two connected components: the connected component of the origin of the group law, given by $x \geq 2$, and the component given by $x^2 \leq 3$.

The quaternion algebra $(x - 2, -1)$ over $\mathbf{Q}(E)$ comes from a (unique) Azumaya algebra over E, which is denoted by A. If M is a p-adic or a real point of E, then the value of A at M is either zero or the unique element of $\operatorname{Br}\mathbf{Q}_v$ of order 2.

An application of Magma (from the School of Mathematics and Statistics at the University of Sydney, Australia) shows that $E(\mathbf{Q}) = \{0, (2, 0)\}$, but in what follows we need only the following statement.

LEMMA 3.15

For any prime p, and for any point $M_p \in E(\mathbf{Q}_p)$, the value $A(M_p)$ is zero. The sum $\sum_v A(M_v)$, taken over all places v of $\mathbf{Q}$, is zero if and only if $M_\mathbf{R}$ is in the connected component of $0 \in E(\mathbf{R})$. In particular, $E(\mathbf{Q})$ is contained in the connected component of $0 \in E(\mathbf{R})$.

Proof

We first prove that A takes only trivial values on $\mathbf{Q}_p$-points of E for any prime p. It is enough to compute the values of A at the points $M_p = (x, y)$ such that $xy \neq 0$. Indeed, since A is an Azumaya algebra over E, for each place v of $\mathbf{Q}$ the map $E(\mathbf{Q}_v) \to \mathbf{Z}/2$ given by evaluation of A is continuous, and for any nonempty Zariski open set U of E, $U(\mathbf{Q}_v)$ is dense in $E(\mathbf{Q}_v)$. Let $K = \mathbf{Q}(\sqrt{-1})$.

Let p be an odd prime. If p splits in K (i.e., if $p \equiv 1 \bmod 4$), then -1 is a square in $\mathbf{Q}_p$ and the assertion is trivial. If p is inert in K (i.e., if $p \equiv 3 \bmod 4$), then $\alpha \in \mathbf{Q}_p^*$ is a norm from K_p, which is equivalent to $(\alpha, -1) = 0 \in \operatorname{Br}\mathbf{Q}_p$, if and only if $v_p(\alpha)$ is even. If $v_p(x) < 0$, then $2v_p(y) = v_p((x^2-3)(x-2)) = 3v_p(x)$. Hence $v_p(x)$ is even, and then $v_p(x-2)$ is even, and so $(x-2, -1) = 0 \in \operatorname{Br}\mathbf{Q}_p$. Assume that $v_p(x-2) \geq 0$. If $v_p(x-2) > 0$, then $v_p(x^2-3) = 0$. Hence $2v_p(y) = v_p(x-2)$, so that $v_p(x-2)$ is even, and we conclude as before.

Let $p = 2$. Write $x = u/v$ with $u \in \mathbf{Z}_2$ and $v \in \mathbf{Z}_2$, not both divisible by 2. In $\mathbf{Z}_2$, we have a relation

$$z^2 = (u^2 - 3v^2)(uv - 2v^2) \neq 0. \tag{18}$$

If $(u, v) \equiv (0, 1)$ or $(1, 0) \bmod 2$, then $u^2 - 3v^2 \equiv 1 \bmod 4$. In both cases, we find $(u^2 - 3v^2, -1) = 0 \in \operatorname{Br}\mathbf{Q}_2$. From (18), we conclude that $(x-2, -1) = 0 \in \operatorname{Br}\mathbf{Q}_2$. It remains to consider the case where $(u, v) \equiv (1, 1) \bmod 2$. Write $x = 1 + 2n$ with $n \in \mathbf{Z}_2$. Then $x - 2 = -1 + 2n$ and $x^2 - 3 = -2 + 4n + 4n^2$. Thus $(x-2)(x^2-3) = 2 + 4m$ for some $m \in \mathbf{Z}_2$, and this cannot be a square. So there are no such points (x, y).

Finally, if $(x, y) \in E(\mathbf{R})$, $y \neq 0$, then $(x-2, -1)_\mathbf{R} = 0$ is equivalent to $x > 2$. Using reciprocity, we obtain the statement about $E(\mathbf{Q})$. □

Let $f : E' \to E$ be the unramified double covering given by $u^2 = x - 2$. The curve E' has a $\mathbf{Q}$-point above zero; choosing it for the origin of the group law on E' turns f into an isogeny of degree 2. We note that $f(E'(\mathbf{R}))$ is the connected component of zero of $E(\mathbf{R})$, so that $E(\mathbf{R})/f(E'(\mathbf{R})) = \mathbf{Z}/2$.

Let D be the Hamilton quaternions. The group $L = \mathrm{SL}_1(D)$ is a $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$-form of SL_2; in particular, it is semisimple simply connected with centre $\{\pm 1\}$. Define $G = (\mathrm{SL}_1(D) \times E')/(\mathbf{Z}/2)$, where $\mathbf{Z}/2$ is generated by $(-1, P)$, $P \in E'(\mathbf{Q})$, $f(P) = 0$, $P \neq 0$. We obtain a commutative diagram of extensions of algebraic groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{SL}_1(D) & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \end{array} \tag{19}$$

This gives rise to the following commutative diagram of pointed sets

$$\begin{array}{ccccc} E(\mathbf{Q}) & \longrightarrow & \mathrm{H}^1(\mathbf{Q}, \mathbf{Z}/2) & \longrightarrow & \mathrm{H}^1(\mathbf{Q}, E') \\ \| & & \downarrow & & \downarrow \\ E(\mathbf{Q}) & \longrightarrow & \mathrm{H}^1\big(\mathbf{Q}, \mathrm{SL}_1(D)\big) & \longrightarrow & \mathrm{H}^1(\mathbf{Q}, G) \end{array} \tag{20}$$

and the compatible diagrams with $\mathbf{Q}_p$ or $\mathbf{R}$ in place of $\mathbf{Q}$.

We have the canonical isomorphisms

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{Z}/2) = \mathbf{Q}^*/\mathbf{Q}^{*2}, \qquad \mathrm{H}^1(\mathbf{R}, \mathbf{Z}/2) = \mathbf{R}^*/\mathbf{R}_{>0},$$

$$\mathrm{H}^1\big(\mathbf{Q}_p, \mathrm{SL}_1(D)\big) = \mathbf{Q}_p^*/\mathrm{Nrd}\big((D \otimes_{\mathbf{Q}} \mathbf{Q}_p)^*\big) = 1,$$
$$\mathrm{H}^1\big(\mathbf{Q}, \mathrm{SL}_1(D)\big) = \mathrm{H}^1\big(\mathbf{R}, \mathrm{SL}_1(D)\big) = \mathbf{R}^*/\mathbf{R}_{>0}.$$

The map $\mathbf{Z}/2 \to \mathrm{SL}_1(D)$ induces a surjection $\mathbf{Q}^*/\mathbf{Q}^{*2} \to \mathrm{H}^1(\mathbf{Q}, \mathrm{SL}_1(D))$, which itself induces a bijection $\{\pm 1\} = \mathrm{H}^1(\mathbf{Q}, \mathrm{SL}_1(D))$.

In the above diagrams, the map $E(\mathbf{Q}) \to \mathbf{Q}^*/\mathbf{Q}^{*2}$ on the affine open set of E defined by $x - 2 \neq 0$ is given by evaluation of the function $x - 2$. As one easily checks, the value on the point at infinity is 1, and the value on the point $x = 2$ is the value taken by $x^2 - 3$, namely, 1. The same statement holds over any field extension of $\mathbf{Q}$.

PROPOSITION 3.16

Let $G/\mathbf{Q}$ be the algebraic group defined above. Let X be a torsor of G whose class $\xi \in \mathrm{H}^1(\mathbf{Q}, G)$ is the image of $-1 \in \mathrm{H}^1(\mathbf{Q}, \mathbf{Z}/2)$ under the map

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{Z}/2) \to \mathrm{H}^1(\mathbf{Q}, G).$$

Then $ob(X) = 0$, $X(\mathbb{A}_{\mathbf{Q}})^{Б} = X(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$, but $X(\mathbf{Q}) = \emptyset$.

Let X^c be a smooth compactification of X. One has $X^c(\mathbb{A}_{\mathbf{Q}})^{Б} = X^c(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$ and $X^c(\mathbb{A}_{\mathbf{Q}})^{\mathrm{Br}_1 X^c} = \emptyset$.

Proof

We use the commutativity and functoriality of the above diagrams. From $\mathrm{H}^1(\mathbf{Q}_p, \mathrm{SL}_1(D)) = 1$, we deduce that the class of $\xi \in \mathrm{H}^1(\mathbf{Q}, G)$ has trivial image in

$\mathrm{H}^1(\mathbf{Q}_p, G)$. From the fact that $E(\mathbf{R}) \to \mathrm{H}^1(\mathbf{R}, \mathbf{Z}/2)$ is onto, we deduce that the class of $\xi \in \mathrm{H}^1(\mathbf{Q}, G)$ has trivial image in $\mathrm{H}^1(\mathbf{R}, G)$. Thus $X(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$.

Next, assume that the image of the class of $-1 \in \mathrm{H}^1(\mathbf{Q}, \mathrm{SL}_1(D))$ in $\mathrm{H}^1(\mathbf{Q}, G)$ is trivial. Then the image of that class in $\mathrm{H}^1(\mathbf{Q}, \mathrm{SL}_1(D))$ comes from $E(\mathbf{Q})$. Restricting to the cohomology over $\mathbf{R}$, we see that the class of -1 in $\mathrm{H}^1(\mathbf{R}, \mathrm{SL}_1(D))$, which is nontrivial, comes from the image of $E(\mathbf{Q})$ in $E(\mathbf{R})$. But $E(\mathbf{Q}) \subset f(E'(\mathbf{R}))$ (see Lem. 3.15), so this is not possible. Thus X is a nontrivial torsor of G, so that $X(\mathbf{Q}) = \emptyset$.

Given the torsor X over $\mathbf{Q}$ under the group G, we may consider the quotient $Y = X/\mathrm{SL}_1(D)$ of X under the action of $\mathrm{SL}_1(D) \subset G$. This is a torsor over $\mathbf{Q}$ under E, whose class in $\mathrm{H}^1(\mathbf{Q}, E)$ is the image of ξ under $\mathrm{H}^1(\mathbf{Q}, G) \to \mathrm{H}^1(\mathbf{Q}, E)$. The projection map $X \to Y$ makes X into a torsor under $\mathrm{SL}_1(D)$. Since ξ comes from $\mathrm{H}^1(\mathbf{Q}, \mathbf{Z}/2)$, the above diagram shows that the class of Y in $\mathrm{H}^1(\mathbf{Q}, E)$ is trivial. We may thus identify $Y = E$. All in all, we see that X is a torsor over E under $\mathrm{SL}_1(D)$.

This argument shows that an open set of X is isomorphic to the affine variety given by the system of equations

$$y^2 = (x^2 - 3)(x - 2) \neq 0, \qquad 2 - x = u^2 + v^2 + w^2 + t^2.$$

Let $\mathfrak{g} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The projection map $X \to E$ induces a Galois equivariant map from the 2-extension of continuous discrete $\mathfrak{g}$-modules

$$1 \to \overline{\mathbf{Q}}[E]^* \to \overline{\mathbf{Q}}(E)^* \to \mathrm{Div}\,\overline{E} \to \mathrm{Pic}\,\overline{E} \to 0$$

to the 2-extension

$$1 \to \overline{\mathbf{Q}}[X]^* \to \overline{\mathbf{Q}}(X)^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0.$$

Over $\overline{\mathbf{Q}}$, the projection $\overline{X} \to \overline{E}$ makes $\overline{X}$ into an SL_2-torsor over $\overline{E}$. Any such torsor is locally trivial for the Zariski topology. Any invertible function on SL_2 is constant, and the Picard group of the simply connected group SL_2 is trivial. From this, we deduce that the maps $\overline{\mathbf{Q}}^* \to \overline{\mathbf{Q}}[E]^* \to \overline{\mathbf{Q}}[X]^*$ and $\mathrm{Pic}\,\overline{E} \to \mathrm{Pic}\,\overline{X}$ are isomorphisms. Pullback from $\overline{E}$ to $\overline{X}$ thus maps the 2-extension

$$1 \to \overline{\mathbf{Q}}^* \to \overline{\mathbf{Q}}(E)^* \to \mathrm{Div}\,\overline{E} \to \mathrm{Pic}\,\overline{E} \to 0$$

to the 2-extension

$$1 \to \overline{\mathbf{Q}}^* \to \overline{\mathbf{Q}}(X)^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0,$$

the map $\mathrm{Pic}\,\overline{E} \to \mathrm{Pic}\,\overline{X}$ being an isomorphism. We have $E(\mathbf{Q}) \neq \emptyset$; hence $ob(E) = 0$. Thus the class of the first extension is trivial; hence so is the class of the second extension. This shows that $ob(X) = 0$.

We now have $ob(X^c) = ob(X) = 0$. Theorem 2.13 then implies that $X(\mathbb{A}_{\mathbf{Q}})^{\text{Б}} = X(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$. It also implies that $X^c(\mathbb{A}_{\mathbf{Q}})^{\text{Б}} = X^c(\mathbb{A}_{\mathbf{Q}}) \neq \emptyset$. This finishes the proof of the proposition. □

Remark. The computation in Lemma 3.15 shows that the counterexample to the Hasse principle on X is due to the Brauer-Manin obstruction given by $\pi^* A \in \operatorname{Br} X$. The class $A \in \operatorname{Br} X$ comes from $\operatorname{Br} E = \operatorname{Br}_1 E$; hence it lies in $\operatorname{Br}_1 X^c$. Hence $X^c(\mathbb{A}_{\mathbf{Q}})^{\operatorname{Br}_1 X^c} = \emptyset$. This is in accordance with a result of Harari [24], which we extend in the appendix.

Appendix. The Brauer-Manin obstruction for homogeneous spaces

Let k be a number field. We denote by Ω the set of all places of k, and we denote by Ω_r the set of all real places of k. If $S \subset \Omega$, we set $k_S = \prod_{v \in S} k_v$. If X is a k-variety, we have $X(k_S) = \prod_{v \in S} X(k_v)$. In particular, $X(k_\Omega) = \prod_{v \in \Omega} X(k_v)$.

For a connected k-group G, we write $G^{\text{ab}} := G/G^{\text{lin}}$; it is the biggest quotient of G which is an abelian variety.

THEOREM A.1
Let G be a connected algebraic group over a number field k. Let X be a homogeneous space of G such that the stabilizers of the geometric points of X are connected. Let X^c be a smooth compactification of X. Assume that a point $x_\Omega = (x_v)_{v \in \Omega} \in X(k_\Omega)$ is orthogonal to $\operatorname{Br}_1 X^c$ *with respect to the Brauer-Manin pairing. Assume that the Tate-Shafarevich group of the maximal abelian variety quotient G^{ab} of G is finite. Then, for any finite set S of* nonarchimedean *places of k and any open neighbourhood $\mathcal{U}_S$ of $x_S = (x_v)_{v \in S}$ in $X(k_S)$, there exists a rational point $x_0 \in X(k)$ whose diagonal image in $\prod_{v \in S} X(k_v)$ lies in $\mathcal{U}_S$. Moreover, we can ensure that, for each archimedean place v, the points x_0 and x_v lie in the same connected component of $X(k_v)$.*

This theorem generalizes a recent result of Harari [24, Th. 1.1]), who considers torsors under a connected algebraic group G. In the extreme case when G is an abelian variety, our result is due to Manin [28] and Wang [48]. In the other extreme case when G is a linear group, this result (including approximation at archimedean places) was obtained in [3, Cor. 2.5]. In the general case, a proof by simple dévissage in order to reduce the assertion to these two extreme cases does not work. Our method of proof uses the reductions and constructions of Sections 3.1 and 3.3 in order to reduce the assertion to the case when X is a k-torsor under a semiabelian variety (treated in [24]) and to the Hasse principle and weak approximation for a homogeneous space of a simply connected semisimple group with connected, character-free geometric stabilizers (results obtained in [2], [1]; see also [12]).

The proof of Theorem A.1 occupies the rest of the appendix.

Proof
Let X be a smooth, geometrically integral k-variety over a number field k. The Brauer-Manin pairing

$$X(k_\Omega) \times \mathrm{Br}_1 X^c \to \mathbf{Q}/\mathbf{Z}$$

defines a map

$$m_X \colon X(k_\Omega) \to (\mathrm{Br}_1 X^c)^D,$$

where $(\mathrm{Br}_1 X^c)^D = \mathrm{Hom}(\mathrm{Br}_1 X^c, \mathbf{Q}/\mathbf{Z})$. By the birational invariance of the Brauer group (see [22]), this map does not depend on the choice of the smooth compactification X^c. If $\varphi \colon X \to Y$ is a morphism of smooth, geometrically integral k-varieties, then by Hironaka's theorem one can construct smooth compactifications Y^c of Y and X^c of X such that φ extends to a morphism $\varphi^c \colon X^c \to Y^c$. The following diagram then commutes:

$$\begin{array}{ccc} X(k_\Omega) & \xrightarrow{m_X} & (\mathrm{Br}_1 X^c)^D \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_*} \\ Y(k_\Omega) & \xrightarrow{m_Y} & (\mathrm{Br}_1 Y^c)^D \end{array}$$

In particular, if $x_\Omega \in X(k_\Omega)$ is a point such that $m_X(x_\Omega) = 0$, and if we define the point $y_\Omega = \varphi(x_\Omega) \in Y(k_\Omega)$, then $m_Y(y_\Omega) = 0$.

Let $x_\Omega \in X(k_\Omega)$ be a point, let S be a finite set of nonarchimedean places of k, and let $\mathcal{U}_{X,S}$ be an open neighbourhood of the S-part x_S of x_Ω. For $v \in \Omega_r$, we denote by $\mathcal{U}_{X,v}$ the connected component of x_v. We set $\mathcal{U}_{X,r} = \prod_{v \in \Omega_r} \mathcal{U}_{X,v}$. We set $\Sigma = S \cup \Omega_r$ and

$$\mathcal{U}_{X,\Sigma} = \mathcal{U}_{X,S} \times \mathcal{U}_{X,r} \subset X(k_\Sigma).$$

Then $\mathcal{U}_{X,\Sigma}$ is an open neighbourhood of x_Σ. We say that $\mathcal{U}_{X,\Sigma}$ is the *special neighbourhood of x_Σ defined by $\mathcal{U}_{X,S}$*.

For the sake of the argument, it is convenient to introduce property (P):

(P) *For any point $x_\Omega \in X(k_\Omega)$ such that $m_X(x_\Omega) = 0$, for any finite set S of nonarchimedean places of k, and for any open neighbourhood $\mathcal{U}_{X,S}$ of x_S, there exists a k-point $x_0 \in X(k) \cap \mathcal{U}_{X,\Sigma}$, where $\mathcal{U}_{X,\Sigma}$ is the special neighbourhood of x_Σ defined by $\mathcal{U}_{X,S}$.*

Theorem A.1 precisely says that property (P) holds for any X as in the theorem.

We need the following lemmas.

LEMMA A.2
Let $\psi: G \to G'$ be a surjective homomorphism of $\mathbf{R}$-groups. Let X be a homogeneous G-variety, and let X' be a homogeneous G'-variety. Let $\varphi: X \to X'$ be a ψ-equivariant morphism. Let $x \in X(\mathbf{R})$, and set $x' = \varphi(x) \in X'(\mathbf{R})$. Then φ takes the connected component of x in $X(\mathbf{R})$ onto the connected component of x' in $X'(\mathbf{R})$.

Proof
Consider the morphism $\lambda_x: G \to X$ defined by $g \mapsto xg$ for $g \in G$. The morphism λ_x is smooth; hence the map $\lambda_x: G(\mathbf{R}) \to X(\mathbf{R})$ is open. We see that the orbit $xG(\mathbf{R})^0$ is open, where $G(\mathbf{R})^0$ is the connected component of 1 in $G(\mathbf{R})$. Clearly, $xG(\mathbf{R})^0$ is connected. Since all the other orbits of $G(\mathbf{R})^0$ are also open, we see that our orbit $xG(\mathbf{R})^0$ is closed; hence it is the connected component of x in $X(\mathbf{R})$. Thus we have proved that the connected components in $X(\mathbf{R})$ are orbits of $G(\mathbf{R})^0$. Similarly, the connected components in $X'(\mathbf{R})$ are orbits of $G'(\mathbf{R})^0$. Consider the action of G on G' by $g' \cdot g = g'\psi(g)$, where $g' \in G',\ g \in G$. By what has been proved, the connected component $G'(\mathbf{R})^0$ of 1 in $G'(\mathbf{R})$ is the $G(\mathbf{R})^0$-orbit of 1. Thus $G'(\mathbf{R})^0 = \psi(G(\mathbf{R})^0)$. Together with the formula $x'\psi(g) = \varphi(xg)$, this shows that φ maps a $G(\mathbf{R})^0$-orbit in $X(\mathbf{R})$ onto a $G'(\mathbf{R})^0$-orbit in $X'(\mathbf{R})$. Thus φ maps the connected component of $x \in X(\mathbf{R})$ onto the connected component of $\varphi(x) \in X'(\mathbf{R})$. □

The following lemma goes back to Cassels and Tate.

LEMMA A.3
Let $\psi: A \to A'$ be a surjective homomorphism of abelian varieties over a number field k. If $Ш(A)$ is finite, then $Ш(A')$ is also finite.

Proof
By Poincaré's complete reducibility theorem (cf. [34, Chap. IV, Sec. 19, Th. 1, p. 173]), there exists an abelian variety A'' over k such that A is isogenous to $A' \times A''$. By [32, Chap. I, proof of Lem. 7.1], it follows that $Ш(A' \times A'')$ is finite. Thus $Ш(A')$ is finite. □

For the sake of completeness, let us give a proof of the following well-known result.

LEMMA A.4
Let $\varphi: Z \to X$ be a torsor under a quasi-trivial torus P, where Z and X are smooth k-varieties over a field k of characteristic zero. Then there is an induced homomorphism $\varphi^*: \mathrm{Br}_1(X^c) \to \mathrm{Br}_1(Z^c)$, and that homomorphism is an isomorphism.

Proof
Let Y be a dense open set of a smooth, proper, geometrically integral variety Y^c. Let $k(Y)$ be the function field of Y. By well-known results of Grothendieck [22], the

morphisms $\operatorname{Spec} k(Y) \to Y \to Y^c$ induce injections $\operatorname{Br} Y^c \subset \operatorname{Br} Y \subset \operatorname{Br} k(Y)$. Let Z, Z^c, X, X^c be as above. By properness of X^c and smoothness of Z^c, the projection morphism $\varphi : Z \to X$ extends to a morphism $\varphi : W \to X^c$, where $W \subset Z^c$ is an open set that contains all points of codimension 1 of Z^c. We thus have a natural map $\varphi^* : \operatorname{Br} X^c \to \operatorname{Br} Z$. By the purity theorem for the Brauer group (see [22], [5, Sec. 3.4]), the restriction map $\operatorname{Br} Z^c \to \operatorname{Br} W$ is an isomorphism. We thus have a homomorphism $\varphi^* : \operatorname{Br} X^c \to \operatorname{Br} Z^c$. The map $\varphi^* : \operatorname{Br} X^c \to \operatorname{Br} Z^c$ is induced by the map $\operatorname{Br} k(X) \to \operatorname{Br} k(Z)$. It is none other than the natural map of unramified cohomology groups $\operatorname{Br}_{\mathrm{nr}}(k(X)/k) \to \operatorname{Br}_{\mathrm{nr}}(k(Z)/k)$ (see [5, Secs. 2.2.1, 2.2.2; Prop. 4.2.3(a)]).

Since P is a quasi-trivial torus over any field F containing k, Shapiro's lemma and Hilbert's theorem 90 yield $\mathrm{H}^1_{\text{ét}}(k(X), P) = 0$. The generic fibre of $Z \to X$ is thus $k(X)$-isomorphic to $P \times_k k(X)$. Since the quasi-trivial torus P as a k-variety is an open set of affine space over k, we see that the field extension $k(Z)/k(X)$ is purely transcendental. From [5, Th. 4.1.5], we get that the map $\varphi^* : \operatorname{Br}_{\mathrm{nr}} k(X) \to \operatorname{Br}_{\mathrm{nr}} k(Z)$ is an isomorphism. Thus $\varphi^* : \operatorname{Br} X^c \to \operatorname{Br} Z^c$ is an isomorphism (use [5, Prop. 4.2.3(a)]). Similarly, $\varphi^* : \operatorname{Br} \overline{X}^c \to \operatorname{Br} \overline{Z}^c$ is an isomorphism. By the very definition of Br_1, we conclude that there is an induced map $\varphi^* : \operatorname{Br}_1 X^c \to \operatorname{Br}_1 Z^c$ and that this map is an isomorphism. □

We now start proving Theorem A.1. The proof is similar to that of Theorem 3.3.

First reduction

Let X and G be as in the theorem. We write G^{u} for L^{u}, where $L = G^{\mathrm{lin}}$. Set $G' = G/G^{\mathrm{u}}$, $Y = X/G^{\mathrm{u}}$. We have a canonical smooth morphism $\varphi \colon X \to Y$. Then Y is a homogeneous space of G' with connected geometrical stabilizers. We have $(G')^{\mathrm{lin}} = G^{\mathrm{lin}}/G^{\mathrm{u}}$; hence $(G')^{\mathrm{u}} = 1$. We have $(G')^{\mathrm{ab}} = G^{\mathrm{ab}}$; hence $Ш((G')^{\mathrm{ab}})$ is finite.

Assume that Y has property (P). We prove that X has this property. Let $x_\Omega \in X(k_\Omega)$ be a point such that $m_X(x_\Omega) = 0$. Set $y_\Omega = \varphi(x_\Omega) \in Y(k_\Omega)$. Since $m_X(x_\Omega) = 0$, we see that $m_Y(y_\Omega) = 0$. Let S and $\mathcal{U}_{X,S}$ be as in (P). Set $\mathcal{U}_{Y,S} = \varphi(\mathcal{U}_{X,S}) \subset Y(k_S)$. Since the morphism $\varphi \colon X \to Y$ is smooth, the map $\varphi \colon X(k_S) \to Y(k_S)$ is open; hence $\mathcal{U}_{Y,S}$ is open in $Y(k_S)$. Set $\Sigma = S \cup \Omega_r$. Let $\mathcal{U}_{Y,\Sigma}$ denote the special open neighbourhood of y_Σ defined by $\mathcal{U}_{Y,S}$. For each $v \in \Omega_r$, let $\mathcal{U}_{X,v}$ denote the connected component of x_v in $X(k_v)$. By Lemma A.2, for each $v \in \Omega_r$ the set $\varphi(\mathcal{U}_{X,v})$ is the connected component of y_v in $Y(k_v)$. Thus $\mathcal{U}_{Y,\Sigma} = \varphi(\mathcal{U}_{X,\Sigma})$. Since Y has property (P), there exists a k-point $y_0 \in Y(k) \cap \mathcal{U}_{Y,\Sigma}$.

Let X_{y_0} denote the fibre of X over y_0. It is a homogeneous space of the unipotent group G^{u}. By [3, Lem. 3.1], the k-variety X_{y_0} has a k-point and has the weak approximation property. Consider the set $\mathcal{V}_\Sigma := X_{y_0}(k_\Sigma) \cap \mathcal{U}_{X,\Sigma}$; it is open in $X_{y_0}(k_\Sigma)$.

Since $y_0 \in \varphi(\mathcal{U}_{X,\Sigma})$, the set $\mathcal{V}_\Sigma$ is nonempty. Since X_{y_0} has the weak approximation property, there is a point $x_0 \in X_{y_0}(k) \cap \mathcal{V}_\Sigma$. Clearly, $x_0 \in X(k) \cap \mathcal{U}_{X,\Sigma}$. Thus X has property (P). Thus, in the proof of Theorem A.1, we may assume that $G^{\mathrm{u}} = 1$.

Second reduction

By Proposition 3.1, we may regard X as a homogeneous space of another connected group G' such that $(G')^{\mathrm{der}}$ is semisimple simply connected, and the stabilizers of the geometric points of X in G' are linear and connected. It follows from the construction in the proof of Proposition 3.1 that there is a surjective homomorphism $G^{\mathrm{ab}} \to (G')^{\mathrm{ab}}$. Since, by assumption, $\text{Ш}(G^{\mathrm{ab}})$ is finite, we obtain from Lemma A.3 that $\text{Ш}((G')^{\mathrm{ab}})$ is finite. Thus if Theorem A.1 holds for the pair (G', X), then it holds for (G, X). We see that in the proof of Theorem A.1 we may assume that G^{lin} is reductive, that G^{der} is semisimple simply connected, and that the stabilizers of the geometric points of X in G are linear and connected.

Relaxing the assumptions

To prove Theorem A.1, it is enough to prove the following result. We write G^{ss} for L^{ss}, where $L = G^{\mathrm{lin}}$. The notation $\overline{H}_1$ was defined in Section 3.1.

THEOREM A.5

Let k be a number field, let G be a connected k-group, and let X be a homogeneous space of G with geometric stabilizer $\overline{H}$. Assume that

(i) $G^{\mathrm{u}} = \{1\}$*;*

(ii) $\overline{H} \subset \overline{G}^{\mathrm{lin}}$*;*

(iii) G^{ss} *is simply connected;*

(iv) $\overline{H}_1$ *is connected and has no nontrivial characters (e.g., $\overline{H}$ is connected);*

(v) $\text{Ш}(G^{\mathrm{ab}})$ *is finite.*

Then X has property (P).

Proof

Recall that the homogeneous space X defines a k-form of $\overline{H}^{\mathrm{mult}}$ which we denote by M (see [3, Sec. 4.1]), and recall that there is a natural homomorphism $M \to G^{\mathrm{sab}}$. We first prove a special case of Theorem A.5.

PROPOSITION A.6

With the hypotheses of Theorem A.5, assume that M injects *into G^{sab} (i.e., assume that $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$). Then X has property (P).*

Proof
Set $Y = X/G^{\mathrm{ss}}$. Then Y is a homogeneous space of the semiabelian variety G^{sab}; hence it is a torsor of some semiabelian variety G'. We have $(G')^{\mathrm{ab}} = G^{\mathrm{ab}}$; hence $Ш((G')^{\mathrm{ab}})$ is finite. We have a canonical smooth morphism $\varphi\colon X \to Y$.

Let $x_\Omega \in X(k_\Omega)$ be a point such that $m_X(x_\Omega) = 0$. Let S, $\mathcal{U}_{X,S}$, and $\mathcal{U}_{X,\Sigma}$ be as in (P). Set $y_\Omega = \varphi(x_\Omega) \in Y(k_\Omega)$. Since $m_X(x_\Omega) = 0$, we see that $m_Y(y_\Omega) = 0$. As in the first reduction, we define $\mathcal{U}_{Y,S} := \varphi(\mathcal{U}_{X,S})$, we construct the corresponding special open neighbourhood $\mathcal{U}_{Y,\Sigma}$ of y_Σ, and we prove that $\mathcal{U}_{Y,\Sigma} = \varphi(\mathcal{U}_{X,\Sigma})$. Now, since Y is a torsor under the semiabelian variety G' whose associated abelian variety $(G')^{\mathrm{ab}}$ has finite Tate-Shafarevich group, by the theorem of Harari [24], the variety Y has property (P). It follows that there exists a k-point $y_0 \in Y(k) \cap \mathcal{U}_{Y,\Sigma}$.

Let X_{y_0} denote the fibre of X over y_0. Consider the set $\mathcal{V}_\Sigma := X_{y_0}(k_\Sigma) \cap \mathcal{U}_{X,\Sigma}$, which is open in $X_{y_0}(k_\Sigma)$. Since $y_0 \in \varphi(\mathcal{U}_{X,\Sigma})$, the set $\mathcal{V}_\Sigma$ is nonempty. In particular, $X_{y_0}(k_v) \neq \emptyset$ for any $v \in \Omega_r$. The variety X_{y_0} is a homogeneous space of G^{ss} with geometric stabilizer $\overline{H} \cap \overline{G}^{\mathrm{ss}} = \overline{H}_1$. The group G^{ss} is semisimple simply connected by assumption (iii) of Theorem A.5. The group $\overline{H}_1$ is connected and has no nontrivial characters by assumption (iv) of Theorem A.5. By [2, Cor. 7.4], the fact that X_{y_0} has points in all real completions of k is enough to ensure that X_{y_0} has a k-point. By [1, Ths. 1.1, 1.4], [12], the variety X_{y_0} has the weak approximation property, and therefore there is a point $x_0 \in X_{y_0}(k) \cap \mathcal{V}_\Sigma$. Clearly, $x_0 \in X(k) \cap \mathcal{U}_{X,\Sigma}$, which shows that X has property (P). $\square$

Let us resume the proof of Theorem A.5. We construct a quasi-trivial k-torus P, the k-group $F := G \times P$, a homogeneous space Z of F, and a morphism $\pi : Z \to X$ as in the proof of Theorem 3.5. Since (Z, π) is a torsor under the quasi-trivial torus P, by Lemma A.4 the canonical map

$$\pi_*\colon \mathrm{Br}_1(Z^c)^D \to \mathrm{Br}_1(X^c)^D$$

is an isomorphism. We have $F^{\mathrm{ab}} = G^{\mathrm{ab}}$; hence $Ш(F^{\mathrm{ab}})$ is finite.

Let $x_\Omega \in X(k_\Omega)$ be a point, and assume that $m_X(x_\Omega) = 0$. Since $\pi : Z \to X$ is a torsor under a quasi-trivial torus, we can lift x_Ω to some $z_\Omega \in Z(k_\Omega)$. We have $m_X(x_\Omega) = \pi_*(m_Z(z_\Omega))$. Since π_* is an isomorphism, from $m_X(x_\Omega) = 0$ we conclude that $m_Z(z_\Omega) = 0$.

Let S be as above, and let $\mathcal{U}_{X,S} \subset X(k_S)$ be an open neighbourhood of x_S. Let $\mathcal{U}_{X,\Sigma} \subset X(k_\Sigma)$ be the corresponding special neighbourhood of x_Σ. Set

$$\mathcal{U}_{Z,S} = \pi^{-1}(\mathcal{U}_{X,S}) \subset Z(k_S).$$

For $v \in \Omega_r$, let $\mathcal{U}_{Z,v}$ be the connected component of z_v in $Z(k_v)$. By Lemma A.2, $\pi(\mathcal{U}_{Z,v}) = \mathcal{U}_{X,v}$. Set $\mathcal{U}_{Z,r} = \prod_{v\in\Omega_r} \mathcal{U}_{Z,v}$, and set $\mathcal{U}_{Z,\Sigma} = \mathcal{U}_{Z,S} \times \mathcal{U}_{Z,r}$. Then $\mathcal{U}_{Z,\Sigma}$ is a special open neighbourhood of z_Σ, and $\pi(\mathcal{U}_{Z,\Sigma}) = \mathcal{U}_{X,\Sigma}$.

The homogeneous space Z of F satisfies the hypotheses of Proposition A.6, so by this proposition there is a point $z_0 \in Z(k) \cap \mathscr{U}_{Z,\Sigma}$. Set $x_0 = \pi(z_0)$; then $x_0 \in X(k) \cap \mathscr{U}_{X,\Sigma}$. Thus X has property (P). This completes the proof of Theorem A.5. □

This also completes the proof of Theorem A.1. □

Acknowledgments. We are grateful to Mathieu Florence, Joost van Hamel, David Harari, Olivier Wittenberg, and Yuri Zarhin for useful discussions. We also thank the Mathematical Sciences Research Institute, Berkeley, California, where this work was done, for hospitality and support.

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