Elliptic curves

Problem sheet 1

May 1, 2009

1 (a) Show that the set of lines in $\mathbb{P}^2_k$ is in a natural bijection with $\mathbb{P}^2_k$. It is called the dual projective plane.

(b) Show that the set of lines through a given point is identified with $\mathbb{P}^1_k$.

(c) Assume that $\mathrm{char}(k) \neq 2$. Let C be the conic $ax^2 + by^2 + cz^2 = 0$, where $abc \neq 0$. Show that the set of lines that are tangent to C is a curve in the dual plane, and find its equation.

2 (a) Prove that for any four points in $\mathbb{P}^2_k$ such that no three of them are collinear there exists a projective transformation sending the four points to

$$(1:0:0),\ (0:1:0),\ (0:0:1),\ (1:1:1).$$

(b) Find the general equation of a conic through these points.

3 Assume that $\mathrm{char}(k) \neq 2$. For any polynomial $f(x)$ find the singular points

(a) of the affine curve C given by $y^2 = f(x)$,

(b) of the projective closure of C.

(c) Now let $\mathrm{char}(k) = 2$. When is the affine curve $y^2 + y = f(x)$ singular?

4 Prove that no irreducible plane curve of degree 4 has three collinear singular points.

5 Find the dimension of the space of cubics that are singular at a given point P.

6 Let $C \subset \mathbb{P}^2_k$ be the curve given by $x^3 = y^2 z$.

(a) Check that $P = (0:0:1)$ is the only singular point on C.

(b) Check that the map $\phi : \mathbb{P}^1_k \to C$ given by $(t:s) \mapsto (ts^2 : s^3 : t^3)$ is a morphism, and that the inverse map is a morphism on $C \setminus P$. Working from the definition of an isomorphism prove that ϕ defines an isomorphism of $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus \{(1:0)\}$ and $C \setminus \{P\}$.

(c) Prove that this isomorphism identifies the group structure on $\overline{k}$ under addition and the group law on a $C \setminus \{P\}$ defined in the usual way.

(d) If you have done (a), (b) and (c), you may want to explore the same questions for the curve $x^3 - x^2 z - y^2 z = 0$.