Elliptic curves

Problem sheet 2

May 1, 2009

1 Consider the plane curve C_a given by $x^3 + y^3 + z^3 = axyz$, where $a \in k$.

(a) Determine the values of a for which C_a is singular, and find the singular points. (Don't forget that $\mathrm{char}(k)$ can be 2 or 3.)

(b) Find the inflection points of C_a.

(c) Assuming that $\mathrm{char}(k) \neq 2$ or 3, find the Weierstrass form of C_0.

2 Find the inflection points of the curves

(a) $yz^2 = x^3$

(b) $yz^2 = x^3 - xz^2$

3 Let C be the elliptic curve $y^2 = x^3 + 4x$ over $\mathbf{Q}$, and $P = (2, 4)$. Find the coordinates of the points nP for all $n \in \mathbf{Z}$.

4 Prove that the equation $y^2 + y = x^3 - x^2$ defines a non-singular curve $E \subset \mathbb{P}^2_{\mathbf{Q}}$. Determine all the points $P = (x, y)$ in $E(\mathbf{Q})$ such that x, $y \in \mathbf{Z}$, $|x| \leq 1$, $|y| \leq 1$, and the subgroup of $E(\mathbf{Q})$ generated by them.

5 (a) Determine the primes p such that $y^2 + y = x^3 - x$ defines a non-singular curve over a field of characteristic p. Check that 2 and 3 are among these primes.

(b) Let E be the elliptic curve with equation $y^2 + y = x^3 - x$ over the finite field $\mathbf{F}_p$ with p elements, where $p = 2$ or $p = 3$. Find the group of points $E(\mathbf{F}_p)$ in these cases.

(c) Can the group $\mathbf{Z} \times \mathbf{Z}/6 \times \mathbf{Z}/8 \times \mathbf{Z}/10$ be realized as the group of points $E(k)$ for some elliptic curve over some field k?