Lie algebras
Problem Sheet 2.

1. Let $\mathfrak{g}$ be the Lie algebra of affine transformations of the line. Compute the Gram matrix of the Killing form of $\mathfrak{g}$. Check that the kernel of the Killing form of $\mathfrak{g}$ is $\mathfrak{g}'$, so that the Killing form is not identically zero.

2. Prove that the quotient of a Lie algebra $\mathfrak{g}$ by its radical is semisimple.

3. Prove that any simple Lie algebra is semisimple.

4. Prove that $\mathfrak{sl}(n)$ is a simple Lie algebra. (Hint: Let E_{ij}, $i \neq j$, be the matrix with the ij-entry equal to 1, and all the other entries equal to 0. If $\mathfrak{a} \subset \mathfrak{sl}(n)$ is a non-zero ideal, and $a \in \mathfrak{a}$, $a \neq 0$, then $[E_{ij}, a] \in \mathfrak{a}$. Use this to prove that $E_{mn} \in \mathfrak{a}$ for some m and n. Deduce that $\mathfrak{a} = \mathfrak{sl}(n)$.)

5. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{a} \subset \mathfrak{g}$ a semisimple ideal. Prove that there exists an ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of Lie algebras.

6. Let $\mathfrak{g}$ be a Lie algebra of dimension n, and $x \in \mathfrak{g}$. Prove that $\mathrm{ad}(x)$ is nilpotent if and only if $a_n(x) = 1$ is the only non-zero coefficient of the characteristic polynomial $P_x(t)$. (Hint: prove first that a linear transformation $k^n \to k^n$ is nilpotent if and only if its characteristic polynomial is t^n.)

7. Prove that the constant term of the characteristic polynomial $P_x(t)$ of $\mathrm{ad}(x)$ for any $x \in \mathfrak{g}$, is identically zero. Conclude that the rank of $\mathfrak{g}$ is at least 1. (Hint: $[x, x] = 0$ so that x is in the kernel of $\mathrm{ad}(x)$, thus $\mathrm{ad}(x)$ has determinant 0.)