Lie algebras
Problem Sheet 3.

1. What are the root systems of rank 1? Give an example of a semisimple Lie algebra which defines such a root system.

2. Classify the root system of rank 2, and compute their Weyl groups.

3. List all pairs of dual root systems of rank 2.

4. In the following cases prove that R is a root system in V, determine the number of roots in R, find its Weyl group of R and the dual root system of R.

A_n Consider the vector space $\mathbb{R}^{n+1}$ with basis $e_1, \ldots, e_{n+1}$ and the standard scalar product, and the subspace V consisting of the vectors with the zero sum of coordinates. Let R be the set of vectors of the form $e_i - e_j$, $i \neq j$.

B_n Let $V = \mathbb{R}^n$ with basis $e_1, \ldots, e_n$ and the standard scalar product. Let R be the set of vectors of the form $\pm e_i$ or $\pm e_i \pm e_j$, $i \neq j$.

C_n The same V, and the set of vectors of the form $\pm 2e_i$ or $\pm e_i \pm e_j$, $i \neq j$.

D_n The same V, and the set of vectors $\pm e_i \pm e_j$, $i \neq j$.

G_2 The unique exceptional root system of rank 2 is called G_2. Show that it can be identified with the set of integers of norm 1 or 3 in $\mathbb{Q}(\sqrt{-3})$. (Describe the remaining root systems of rank 2 in a similar way.)

F_4 Consider the lattice $L \subset \mathbb{R}^4$ generated by the basis vectors e_i and the vector $(e_1 + e_2 + e_3 + e_4)/2$. Let R be the set of vectors $v \in L$ such that $(v, v) = 1$ or $(v, v) = 2$. This root system is called F_4.

E_8 Consider the lattice $L \subset \mathbb{R}^8$ generated by the basis vectors e_i and the vector $(e_1 + \ldots + e_8)/2$, and let $L_0 \subset L$ be the sublattice consisting of the vectors with even sum of coordinates. Let R be the set of vectors $v \in L'$ such that $(v, v) = 2$. This root system is called E_8.

E_6 and E_7 The intersection of the root system of type E_8 with the linear span of $e_1, \ldots, e_6$ (resp. $e_1, \ldots, e_7$) defines a root system in $\mathbb{R}^6$ (resp. $\mathbb{R}^7$). This root system is called E_6 (resp. E_7).

5. Describe the root system obtained as the intersection of E_8 with the linear span of $e_1, \ldots, e_n$ for $n = 2, 3, 4, 5$.

6. Prove that the following root systems are equivalent: $A_1 \simeq B_1 \simeq C_1$, $C_2 \simeq B_2$, $D_2 \simeq A_1 \times A_1$, $D_3 \simeq A_3$.