BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2009

M1GLA (Solutions to the test)

Geometry and Linear Algebra

- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) **2 marks**

We can take $(-1, 2/3)$ to be a direction vector of the second line. Thus the two lines are parallel if and only if $a = 2/3$.

(ii) **2 marks**

We can take $n = (1, 2, -3)$ to be a normal vector to Π. The length $||n||$ is $\sqrt{14}$. Thus the perpendicular distance is

$$|((0, 1, -1) - (1, 1, 1)).(1, 2, -3)|/\sqrt{14} = 5/\sqrt{14}.$$

(iii) **3 marks**

X is the intersection of the perpendicular bisectors to AB and AC. The first of these is given by $x.(1, 3) = ||(1, 3)||^2/2$, which is $x_1 + 3x_2 = 5$. The second one is $x.(-1, 5) = ||(-1, 5)||^2/2$, which is $-x_1 + 5x_2 = 13$. Solving for x_1 and x_2 we find $X = (-7/4, 9/4)$.

(iv) **3 marks**

$(\pm 3, 0)$, any proof is fine.

(v) **5 marks**

The matrix $\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$ has eigenvalues ± 25. It follows that for $c \neq 0$ we get a hyperbola, whereas if $c = 0$ we get a pair of intersecting lines.

(vi) **5 marks**

$$\det \begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix} = 1 + a^2 + b^2 + c^2 > 0$$

for any real a, b, c. Hence $I + A$ is invertible for any skew-symmetric 3×3-matrix A.

SECTION B

2. (i) **3 marks**

A matrix is in echelon form if the first non-zero entry in the $i+1$-st row stands to the right of the first non-zero entry in the i-th row, for all i. It is also assumed that the all-zero rows (if any) must be at the bottom.

(ii) **3 marks**

(1) Replace row(i) with the sum row$(i) + a$row(j), for any a, $j \neq i$..

(2) Swap two rows.

(3) Multiply a row by a non-zero number.

(iii) **4 marks**

Moving from left to right, we find the first non-zero column. Using operation (2) we ensure that the entry in the first row is non-zero. Using operation (1) we clear that column, i.e. make all the other entries equal to 0. Repeat this procedure to the matrix consisting of the entries which stand below and to the right of the non-zero element. Continue until only the all-zero rows are left.

(iv) **10 marks**

Consider the 4×8-matrix obtained by writing the identity matrix to the right of our matrix. Use elementary row operations to reduce this matrix to I. Then the right hand part of the resulting 4×8-matrix is the inverse. It equals

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix}$$

3. (i) **2 marks**

A square matrix P is orthogonal if $P^T \cdot P = I$.

(ii) **8 marks**

The rows of A are unit vectors orthogonal to each other. Since $\det A = -1$ we can write

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

for some $\alpha \in \mathbb{R}$. The characteristic polynomial is $t^2 - 1$, hence A has two eigenvalues: 1 and -1.

(iii) **10 marks**

By lectures we must have

$$\det \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} = a^3 - 3a + 2 = 0$$

Hence $a = 1$ (double root) or $a = -2$. The value $a = 1$ gives an inconsistent system, but $a = -2$ gives a system with infinitely many solutions $(m, m, m+1)$ for any $m \in \mathbb{R}$.