M1GLA Geometry and Linear Algebra

Exercise Sheet 1

(*not for assessment*)

1. The following axiom system describes the rules of an international espionage network. The undefined terms are *a spy*, *a country* and *to spy*. The axioms are as follows:

(1) There is at least one spy.

(2) For any two different countries, there is exactly one spy who spies for both of them.

(3) Each spy spies for at least two countries.

(4) For any spy, there is at least one country for which it does not spy.

Prove the following statements:

(a) There are at least three countries.

(b) If there are exactly 3 countries, then there are also exactly 3 spies.

(c) If there are exactly 4 countries, then the number of spies is either 4 or 6.

2. Prove Thales's Theorem: if A, B, D are points on a circle with centre C, such that C and D are on the same side of the line AB, then the angle ADB is half the angle ACB. What if D and C are on different sides of the line AB? Deduce that if AB is a diameter of a circle, then ADB is a right angle for any point D on the circumference.

3. Prove the cosine rule for triangles: in a triangle ABC, $c^2 = a^2 + b^2 - 2ab\cos C$.

4. Devise ruler and compass constructions to do the following:

(a) construct an equilateral triangle on a given line segment AB

(b) draw a tangent to a given circle at a given point of the circle

(c) given lengths 1 and x, construct a length $\sqrt{x}$

(d) given a length 1, construct a length $\sqrt{1+\sqrt{2}}$.

5. The members of a club have formed various committees in accordance with the following rules:

(1) for any two members A and B, there is exactly one committee containing both A and B

(2) any two committees have exactly one member in common

(3) each committee has exactly 3 members

(4) there are at least 2 committees.

Prove that the club has exactly 7 members and 7 committees.