M1GLA Geometry and Linear Algebra
Exercise Sheet 10

1. (a) Find the (vector) equation of the line joining the points $(3, 1, 2)$ and $(6, 1, 8)$. Find also the foot and the length of the perpendicular from $(0, 2, 1)$ to this line.

(b) Show that the same line is represented by the following two sets of equations:

$$\frac{x_1 - 6}{3} = \frac{x_2 + 2}{6} = \frac{x_3 - 2}{9}, \quad \text{and } \frac{x_1 - 7}{2} = \frac{x_2}{4} = \frac{x_3 - 5}{6}.$$

(c) Find the equation of the plane normal to $n = (2, -1, 1)$ and containing the point $(-1, 0, 1)$. Find the equation of the line through the point $(1, -2, 1)$ which is perpendicular to this plane. Calculate the perpendicular distance from $(1, -2, 1)$ to the plane.

2. Prove that any two distinct planes meet in either a line or the empty set.

3. Let A, B, C be the points $(1, 2, 3), (2, 3, 5), (4, 3, 1)$ respectively.

(a) Find the area of the triangle ABC.

(b) Find the volume of the parallelepiped with corners A, B, C, D, where D is $(-2, 0, 2)$.

(c) Find α such that the four points A, B, C and $(-2, 0, \alpha)$ are coplanar. For this value of α, find the equation of the plane containing these four points.

4. Let a, b be nonzero vectors in $\mathbb{R}^3$ which are not scalar multiples of each other. Prove that if n is a vector which is perpendicular to both a and b, then n is a scalar multiple of $a \times b$.

5. Find the equations of the following planes:

(i) through the point $(2, -1, 1)$ and normal to the vector $(2, 3, -1)$,

(ii) normal to the vector $(2, 1, -2)$ and at a perpendicular distance of 3 from the point $(1, 2, 3)$.

Find the angle between the planes in (i) and (ii).

6. Show that if x, a, b are vectors in $\mathbb{R}^3$ such that $\alpha x + (x \times a) = b$, where α is a non-zero real number, then

$$x = \frac{1}{(\alpha^2 + ||a||^2)}(\alpha b + \frac{a.b}{\alpha}a + (a \times b)).$$

(*Hint:* Try dotting and then crossing both sides of the equation with a.)

7. Prove the following:

(i) $a \times (b \times (c \times d))) = (b.d)(a \times c) - (b.c)(a \times d)$.

(ii) $(a \times b) \times (c \times d) = (a.(c \times d))b - (b.(c \times d))a$.