M1GLA Geometry and Linear Algebra
Exercise Sheet 3

1. A food packaging company receives daily 6 tons of coffee from America, 7 tons from Africa, and 4 tons from Asia. It can sell blend A, which consists of America:Africa:Asia in the ratios 2:1:1, at a profit of 5 pounds per ton, and can sell blend B, which has corresponding ratios 1:2:1, at a profit of 6 pounds per ton.

If the company prepares daily x_1 tons of blend A and x_2 tons of blend B, write down the inequalities that x_1 and x_2 must satisfy, and sketch the region in the x_1x_2-plane satisfying these inequalities. Find the amount of each blend that the company should prepare daily to maximise the profit from sales of these two blends.

[*You can use the general fact stated in the lectures: a linear function takes its extreme (maximal and minimal) values at the corners of a domain bounded by straight lines. In theory it can be any corner!*]

2. (i) Let L be the line with equation $x_1+2x_2 = 3$. Find the value of α if the distance from the point $(0, \alpha)$ to L is equal to 1.

(ii) Let a, b, c be the three points $(4, 2), (3, 3), (0, 2)$. Find the centre and radius of the circle through a, b, c.

(iii) Consider the same circle as in (ii). Find an equation of its tangent at a.

3. (i) If a, b are points in the plane, prove that the equation of the perpendicular bisector of the line ab is

$$x.(a - b) = \frac{1}{2}(||a||^2 - ||b||^2)$$

(ii) If $\{a, b, c\}$ is a triangle, prove that the perpendicular bisectors of the sides of the triangle meet at a point.

(This point is called the *circumcentre* of the triangle.)

(iii) Find the circumcentre of the triangle $\{a, b, c\}$, where a, b, c are the points $(2, -1)$, $(3, 1)$, $(1, -2)$.

4. Decide which type of conic is given by each of the following equations by reducing the equation to standard form, and sketch the conic in the x_1x_2-plane.

(i) $x_1x_2 = -2$

(ii) $x_1^2 - x_2^2 + 2x_1 + 4x_2 - 3 = 0$

(iii) $3x_1^2 + 6x_1x_2 + 3x_2^2 + 12\sqrt{2}x_1 + 6 = 0$

(iv) $\sqrt{3}x_1x_2 - x_2^2 - 1 = 0$.

5. (i) Prove that for any vectors $x, y, z \in \mathbb{R}^2$,

$$||x + y + z|| \leq ||x|| + ||y|| + ||z||.$$

What can be said when equality holds ?

(ii) Prove the "3-dimensional" version of the Cauchy-Schwarz inequality: if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then

$$|x.y| \leq ||x||\,||y||,$$

where $x.y = x_1y_1 + x_2y_2 + x_3y_3$ and $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.