M1GLA Geometry and Linear Algebra
Exercise Sheet 5

1. Find all solutions of the following systems of linear equations:

(a)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 8 \\ 3x_1 - 6x_2 + 2x_3 &= 18 \\ x_3 - 2x_4 &= 5 \\ 2x_1 - 3x_2 + 3x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ 3x_1 - 8x_2 + 2x_3 &= 5 \\ 3x_1 - 7x_2 + x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_3 + x_4 + x_5 &= 0 \\ 2x_1 - x_2 + x_3 - 3x_4 - x_5 &= 0 \\ 9x_1 - 3x_2 - x_3 - 7x_4 &= 4 \end{aligned}$$

(d)
$$\begin{aligned} x_2 + 2x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

Having found the general solution, give a solution to (c) with $x_2 = -3$.

2. Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ 2x_1 + x_2 + ax_3 &= 1 \\ 3x_1 + x_2 + x_3 &= b \end{aligned}$$

where a and b are real numbers. For which values of a and b does the system have

(i) no solutions ?

(ii) exactly one solution ?

(iii) infinitely many solutions ?

3. Calculate all valid products of the following matrices (i.e. AB, BA, A^2, B^2, etc.):

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Find a formula for A^n (n a positive integer).

5. The football teams A, B, C and D played each of the others twice.

When all the matches had been played, the following facts emerged: (1) the average number of goals per game was 3; (2) A scored twice as many goals as B and C put together; (3) the difference between the numbers of goals scored by A and B was four times the difference between the numbers scored by C and D; (4) D scored the smallest number of goals.

How many goals did each team score ?