M1GLA Geometry and Linear Algebra
Exercise Sheet 6

1. (i) For the following square 2×2-matrices A find the smallest positive integer m such that $A^m = I_2$ (the identity matrix), or show that no such m exists:

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

(ii) Find A^4 for the following matrices A:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. A square $n \times n$-matrix $A = (a_{ij})$ is called *stochastic* if $a_{ij} \geq 0$ for all i and j (all entries are non-negative), and for any j we have $\sum_{i=1}^{n} a_{ij} = 1$ (the sum of entries in every column is 1).

(i) Prove that A is stochastic if and only if for every n-dimensional column vector p satisfying $p_1 + \ldots + p_n = 1$ and $p_i \geq 0$ for any i, the column vector Ap satisfies the same conditions.

(ii) If, moreover, $a_{ij} > 0$ for all i and j, and p is any n-dimensional column vector p satisfying $p_1 + \ldots + p_n = 1$ and $p_i \geq 0$ for any i, then all the coordinates of Ap are positive.

3. Prove that the product of two stochastic matrices of the same size is again a stochastic matrix.

Hint. If you are not sure how to start, do Questions 2 and 3 for matrices of size 2, then for size 3, then guess the proof for the case of an arbitrary n.

4. Find all 2×2-matrices A such that $AB = BA$ for any 2×2-matrix B. Justify your answer (i.e. prove that the matrices you found do commute with any matrix, and that these are the only matrices with this property.)

5. The same question for $n \times n$-matrices.

6. Find all 2×2-matrices A such that $AB = BA$, where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$