M1GLA Geometry and Linear Algebra
Exercise Sheet 9

1. (a) Let A be an invertible square matrix. Prove that $(A^T)^{-1} = (A^{-1})^T$.

(b) Is the inverse of an invertible symmetric matrix necessarily symmetric?

(c) Let A be a $n \times n$-matrix with entries $a_{ii} = 0$ for any i, and $a_{ij} = 1$ for any $i \neq j$. Without evaluating any determinants prove that -1 and $n-1$ are eigenvalues of A.

2. A square matrix A is called skew-symmetric if $A^T = -A$.

(a) Prove that any skew-symmetric 2×2-matrix is of the form $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ for some $a \in \mathbb{R}$. Find the characteristic polynomial and the (complex) eigenvalues of A.

(b) Prove that any skew-symmetric 3×3-matrix is of the form $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Find the characteristic polynomial and the (complex) eigenvalues of A.

(c) Let A be a skew-symmetric matrix, and P an orthogonal matrix. Prove that $P^{-1}AP$ is skew-symmetric.

(d) Is the inverse of an invertible skew-symmetric matrix necessarily skew-symmetric?

3. (a) A and B are square matrices such that $A = B^3$, and A is symmetric. Is B necessarily symmetric?

(b) Reduce the conic $11x_1^2 + 6x_1x_2 + 19x_2^2 = a$ to standard form using the matrix method from the lectures; hence determine its type for any $a \in \mathbb{R}$.

4. (a) Show that any symmetric 2×2-matrix with eigenvalues ±1 is orthogonal.

(b) Let $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Let $A = \begin{pmatrix} 1 & * & * \\ 0 & \zeta & * \\ 0 & 0 & \zeta^2 \end{pmatrix}$, where $*$ denote unspecified complex numbers. Find A^3.

5. Let A be a matrix of any size with real entries.

(a) Let $\lambda \in \mathbb{C}$ be a complex eigenvalue of A. Then $|A - \lambda I| = 0$, so that the system of linear equations $Ax = \lambda x$ has a non-zero solution in $\mathbb{C}^n$, a complex eigenvector $v = (v_1, \dots, v_n)$ with eigenvalue λ. Show that the vector $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$, where bar denotes complex conjugation, is an eigenvector of A with eigenvalue $\bar{\lambda}$.

(b) Observe that $\bar{v}.v > 0$. Deduce that if A is symmetric, then $\lambda \in \mathbb{R}$.

(c) If A is skew-symmetric, then λ is purely imaginary, that is, $\lambda = ix$, $x \in \mathbb{R}$.

(d) If A is orthogonal, then $|\lambda| = 1$, that is, $\lambda = \cos\phi + i\sin\phi$, $\phi \in \mathbb{R}$.