M1GLA Geometry and Linear Algebra, Solutions to Sheet 1

1. (a) There is a spy, by (1). He spies for at least 2 countries by (3), and there is a further country by (4). Hence there are at least 3 countries.

(b) Say the 3 countries are C_1, C_2, C_3. For each pair i, j there is one spy S_{ij} who spies for both C_i and C_j. This gives 3 spies S_{12}, S_{13}, S_{23}. These spies are all different by (4), and there are no further spies by (2). So there are exactly 3 spies.

(c) Let the countries be C_1, C_2, C_3, C_4. If there is no spy spying for more than 2 countries, then the argument of (b) shows that there are exactly 6 spies S_{12}, S_{13}, S_{14}, S_{23}, S_{24}, S_{34}.

Finally suppose there is a spy S spying for 3 countries (can't spy for 4 by (4)); say S spies for C_1, C_2, C_3. For $i = 1, 2, 3$ there are spies S_{i4} spying for C_i, C_4, and these are 3 different spies by (2) (e.g. if S_{14} is the same as S_{24}, then S and S_{14} spy for both C_1 and C_2 contrary to (2)). So there are exactly 4 spies, $S, S_{14}, S_{24}, S_{34}$.

2. Using the theorem that the sum of angles of any triangle is π, the full circle is 2π, and the properties of isosceles triangles, we obtain

$$ACB = 2\pi - ACD - BCD = 2\pi - (\pi - 2CDA) - (\pi - 2CDB) = 2(CDA + CDB) = 2ADB.$$

3. Drop a perpendicular from A to BC, meeting BC at D. Let d be the length of AD, and x the lenth of DC. Applying Pythagoras in the triangles ABD and ACD gives $c^2 = (a-x)^2 + d^2 = (a-x)^2 + b^2 - x^2 = a^2 + b^2 - 2ax = a^2 + b^2 - 2ab\cos C$.

4. (a) Draw circles centre A, radius AB, and centre B, radius AB. Any of the two points C of their intersection is the third corner of an equilateral triangle ABC.

(b) Let C be the centre of the circle and P a point on the circle. Draw any circle with centre at P. Let A and B be the points in which it meets the line CP. Then the perpendicular bisector of AB (constructed as in lectures) is the required tangent.

(c) Big hint given in lectures.

(d) Construct $\sqrt{2}$ as diameter of a square of side 1. Produce to get length $1 + \sqrt{2}$. Then construct the square root of this using (c).

5. Represent the members by points and the committees by lines, so the axioms are: (1) any 2 points lie on 1 line; (2) any 2 lines meet in 1 point; (3) each line has 3 points; (4) there are at least 2 lines. Now argue pictorially that the picture has 7 points and 7 lines.

Such a situation naturally arises as follows. The members are represented by the vectors (x_1, x_2, x_3) with coordinates 0 or 1, not all three equal to 0, and a committee is a set of vectors satisfying the condition $a_1x_1 + a_2x_2 + a_3x_3 = 0$, where a_1, a_2, a_3 are 0 or 1, not all three equal to 0. Here the addition and multiplication are understood in the sense of binary arithmetic ($0\times 0 = 1\times 0 = 0\times 1 = 0$, $1\times 1 = 1$, $0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$).