M1GLA Geometry and Linear Algebra, Solutions to Sheet 10

1. (a) The equation is $x = (3+t, 1, 2+2t)$. The foot of the perpendicular is (α, β, γ), where $(\alpha, \beta, \gamma) = (3+t, 1, 2+2t)$ for some t, and $(\alpha, \beta-2, \gamma-1).(1, 0, 2) = 0$. Solution is $(\alpha, \beta, \gamma) = (2, 1, 0)$. The length of the perpendicular is $\sqrt{6}$.

(b) Both lines are $x = (6, -2, 2) + t(1, 2, 3)$.

(c) Plane is $2x_1 - x_2 + x_3 = -1$. Line is $x = (1+2t, -2-t, 1+t)$. Perpendicular distance is $\sqrt{6}$.

2. Let the planes have equations $ax_1+bx_2+cx_3 = p$ and $dx_1+ex_2+fx_3 = q$. If $(a, b, c) = \lambda(d, e, f)$, then $p \neq \lambda q$ since the planes are distinct. Then these equations are inconsistent and the planes meet in the empty set. Otherwise the two equations reduce to echelon form with one free variable, and the solutions form a line.

3. (a) Area is $\frac{1}{2}||(b-a) \times (c-a)|| = \sqrt{21}$.

(b) Volume is $|det \begin{pmatrix} b-a \\ c-a \\ d-a \end{pmatrix}| = 2$.

(c) The condition that $E = (-2, 0, \alpha)$ lies in the plane of A, B, C is that $(e-a).((c-a) \times (b-a)) = 0$, i.e. $det \begin{pmatrix} e-a \\ c-a \\ b-a \end{pmatrix} = 0$. This gives $\alpha = 1$. Equation is $2x_1 - 4x_2 + x_3 = -3$.

4. If $n = (x_1, x_2, x_3)$, then equations $n.a = n.b = 0$ give two linear equations in x_1, x_2, x_3 which are not multiples of each other. Solutions are multiples of a fixed vector; as $a \times b$ is perpendicular to both a and b, the solutions must be multiples of $a \times b$.

5. (i) $2x_1 + 3x_2 - x_3 = 0$

(ii) Since $(2, 1, -2)$ has length 3, the point where the perpendicular from $(1, 2, 3)$ meets the plane is either $(1, 2, 3) + (2, 1, -2)$ or $(1, 2, 3) - (2, 1, -2)$. Hence the plane is either $2x_1 + x_2 - 2x_3 = 7$ or $2x_1 + x_2 - 2x_3 = -11$.

(iii) Angle between planes = angle between normals $n = (2, 3, -1)$ and $m = (2, 1, -2)$. Using $m.n = ||m||\ ||n|| \cos\theta$, see that the angle is $\cos^{-1}(3/\sqrt{14})$.

6. Dot both sides with a to get $x.a = (b.a)/\alpha$. Cross both sides with a and use $(x \times a) \times a = (x.a)a - (a.a)x = (b.a)a/\alpha = (a.a)x$ to get $\alpha(x \times a) + (b.a)a/\alpha - ||a||^2 x = b \times a$. Combining this with the original equation and eliminating $x \times a$ gives the result.

7. Straightforward applications of the equality $(a \times b) \times c = (a.c)b - (b.c)a$.