M1GLA Geometry and Linear Algebra, Solutions to Sheet 2

1. (i) $L = \{(1,2) + \lambda(-3,3) : \lambda \in \mathbb{R}\}$ (ii) $n = (1,1)$ (iii) $\alpha = -3$ (iv) $(\frac{4}{3}, \frac{5}{3})$

2. (i) The line is $\{a + \lambda(b-a) : \lambda \in \mathbb{R}\}$, which has direction vector $b - a = (-2,1)$, hence normal $(1,2)$. Hence its equation is $(x-a).(1,2) = 0$, which is $x_1 + 2x_2 - 5 = 0$.

(ii) The line has the form $x_1+2x_2+r = 0$ and contains $(0,7)$, so it is $x_1+2x_2-14 = 0$.

(iii) Line is $(x-c).(2,-1) = 0$, which is $2x_1 - x_2 + 7 = 0$.

(iv) Intersection is $(-9/5, 17/5)$.

(v) The perpendicular distance is the distance between c and $(-9/5, 17/5)$, which is $9/\sqrt{5}$.

(vi) If the angle in question is θ, then $(b-a).(c-a) = ||b-a|| \cdot ||c-a|| \cdot \theta$, giving

$$\theta = \cos^{-1}(\frac{7}{\sqrt{130}}).$$

3. (i) Since $b - c$ is a normal to this altitude and a is a point on the altitude, its equation is $x.(b-c) = a.(b-c)$.

(ii) The three altitudes have equations

$$\begin{array}{ll} (1) & x.(b-c) = a.(b-c) \\ (2) & x.(c-a) = b.(c-a) \\ (3) & x.(a-b) = c.(a-b) \end{array}$$

Adding (1) and (2) gives (3). Hence the point of intersection of lines (1) and (2) lies on (3), i.e. the three altitudes meet in a common point.

(iii) For a, b, c as given, two altitudes are $x.(2,-4) = (1,2).(2,-4)$ and $x.(1,-3) = (0,3).(1,-3)$. These meet at $(9,6)$.

4. Let the two lines be $L_1 = \{a+\lambda u : \lambda \in \mathbb{R}\}$ and $L_2 = \{b+\mu v : \mu \in \mathbb{R}\}$. If x is a point of intersection then there are real numbers λ_0, μ_0 such that $x = a + \lambda_0 u = b + \mu_0 v$. This gives two linear equations for λ_0, μ_0:

$$\begin{array}{ll} (1) & \lambda_0 u_1 - \mu_0 v_1 = b_1 - a_1 \\ (2) & \lambda_0 u_2 - \mu_0 v_2 = b_2 - a_2 \end{array}$$

Then $((1) \times v_2) - ((2) \times v_1)$ gives

$$\lambda_0(u_1 v_2 - u_2 v_1) = c$$

(where $c = v_2(b_1 - a_1) - v_1(b_2 - a_2)$). As L_1, L_2 are non-parallel, u, v are not scalar multiples of each other, so $u_1 v_2 - u_2 v_1 \neq 0$. Hence $\lambda_0 = \frac{c}{u_1 v_2 - u_2 v_1}$. Similarly $\mu_0 = \frac{c'}{u_1 v_2 - u_2 v_1}$ where $c' = u_2(b_1 - a_1) - u_1(b_2 - a_2)$. So equations (1), (2) have a unique solution for λ_0, μ_0 and so there is a unique point of intersection $x = a + \lambda_0 u = b + \mu_0 v$.

Note: it is also fine to do this question starting with linear equations for the two lines

5. The first line is $x.(p,q) = -r$, so it has normal (p,q), hence has direction vector $(-q,p)$. Similarly the second line has direction vector $(-q',p')$. So if the lines are perpendicular then $(-q,p).(-q',p') = 0$, i.e. $qq' + pp' = 0$.