M1GLA Geometry and Linear Algebra, Solutions to Sheet 3

1. The inequalities are

$$x_1 \geq 0, x_2 \geq 0, x_1/2 + x_2/4 \leq 6, x_1/4 + x_2/2 \leq 7, x_1/4 + x_2/4 \leq 4.$$

Simplify these to

$$x_1 \geq 0, x_2 \geq 0, 2x_1 + x_2 \leq 24, x_1 + 2x_2 \leq 28, x_1 + x_2 \leq 16.$$

Draw the region and compute its corners: $(0,0)$, $(12,0)$, $(8,8)$, $(4,12)$, $(0,14)$. Check that maximum profit $5x_1 + 6x_2$ over the region occurs at the point $(4,12)$.

2. (i) Let $p = (0,\alpha)$, let $n = (1,2)$, a normal to L, and $a = (1,1)$, a point on L. By lectures we have

$$dist(p,L) = |(p-a).n|/||n|| = |-1+2(\alpha-1)|/\sqrt{5}.$$

To make this 1, we need $\alpha = (3 \pm \sqrt{5})/2$.

(ii) If the centre is $c = (c_1, c_2)$, then the equation of the circle centered at c is

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 = r^2,$$

where r is the radius. If a, b, c lie on this circle, then we have

$$(4 - c_1)^2 + (2 - c_2)^2 = (3 - c_1)^2 + (3 - c_2)^2 = (0 - c_1)^2 + (2 - c_2)^2.$$

First equation yields $2c_1 - 2c_2 = 2$, second gives $6c_1 + 2c_2 = 14$. Solve to get $(c_1, c_2) = (2,1)$. Radius is the distance from centre to any of the points, which is $\sqrt{5}$.

(iii) The tangent at a is normal to the vector $a - c = (2,1)$, so taking $n = (2,1)$, the equation of the tangent is $x.n = a.n$, i.e. $2x_1 + x_2 = 10$.

3. (i) The mid-point of ab is $\frac{1}{2}(a+b)$, so the equation of the perpendicular bisector is $(x - \frac{1}{2}(a+b)).(a-b) = 0$. This works out as $x.(a-b) = \frac{1}{2}(a+b).(a-b) = \frac{1}{2}(||a||^2 - ||b||^2)$.

(ii) The three perpendicular bisector have equations

$$\begin{array}{ll} (1) & x.(a-b) = \frac{1}{2}(||a||^2 - ||b||^2) \\ (2) & x.(b-c) = \frac{1}{2}(||b||^2 - ||c||^2) \\ (3) & x.(c-a) = \frac{1}{2}(||c||^2 - ||a||^2) \end{array}$$

Adding (1) and (2) gives (3). Hence the point of intersection of lines (1) and (2) lies on (3), i.e. the three lines meet in a common point.

(iii) For a, b, c as in the question, the three perpendicular bisectors are $x_1 + 2x_2 = 5/2$, $2x_1 + 3x_2 = 5/2$ and $x_1 + x_2 = 0$. Their point of intersection is $(-5/2, 5/2)$.

4. (i) Rotate axes (anticlockwise) through $\pi/4$. If (y_1, y_2) are the new coordinates, then as in lectures we have $x_1 = (y_1 - y_2)/\sqrt{2}$, $x_2 = (y_1 + y_2)/\sqrt{2}$, and the equation becomes $y_1^2 - y_2^2 = -4$, or $y_2^2/4 - y_1^2/4 = 1$, a hyperbola.

(ii) Translation $y_1 = x_1 + 1$, $y_2 = x_2 - 2$ gives equation $y_1^2 - y_2^2 = 0$, which is a pair of lines, $y_1 = \pm y_2$.

(iii) As in (i) rotate through $\pi/4$ to get $6y_1^2 + 12y_1 - 12y_2 + 6 = 0$. The translation $z_1 = y_1 + 1$, $z_2 = y_2$, gives $y_1^2 - 2y_2 = 0$, a parabola.

(iv) By lectures, the first step in reducing a conic $ax_1^2 + bx_1x_2 + cx_2^2 + ... = 0$ is to rotate through angle θ, where $\tan 2\theta = b/(a-c) = \sqrt{3}$, so $\theta = \pi/6$. This changes coordinates to y_1, y_2, where $x_1 = y_1 \cos\theta - y_2 \sin\theta$, $x_2 = y_1 \sin\theta + y_2 \cos\theta$, whence equation becomes $y_1^2/2 - 3y_2^2/2 = 1$, which is a hyperbola.

5. (i) $||x+y+z|| \leq ||x+y|| + ||z||$ (by the triangle inequality) which is $\leq ||x|| + ||y|| + ||z||$ by the triangle inequality again. Equality holds if and only if x, y, z are collinear and all three vector point in the same direction. PTO for (ii).

(ii) Similar to the proof for $\mathbb{R}^2$ in lectures:

$$|x.y|^2 \leq ||x||^2||y||^2 \Leftrightarrow (x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2),$$

which after expanding and cancelling terms works out as

$$\Leftrightarrow 0 \leq (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + (x_2y_3 - x_3y_2)^2,$$

which is true, hence result.