M1GLA Geometry and Linear Algebra, Solutions to Sheet 4

1. (i) We need to rotate this conic through θ where $tan2\theta = b/(a-c) = \sqrt{3}$, so that $\theta = \pi/6$. This leads to $x_1 = (\sqrt{3}/2)y_1 - (1/2)y_2$, $x_2 = (1/2)y_1 + (\sqrt{3}/2)y_2$. The equation becomes $4y_2^2 + 4y_1 = 0$, that is, $y_2^2 + y_1 = 0$, a parabola.

(ii) Now $tan2\theta = b/(a-c) = 4/3$. Using $tan2\theta = 2tan\theta/(1-tan^2\theta)$ we find that $tan\theta = 1/2$ will do. The $cos\theta = 2/\sqrt{5}$, $sin\theta = 1/\sqrt{5}$. Thus $x_1 = (2/\sqrt{5})y_1 - (1/\sqrt{5})y_2$, $x_2 = (1/\sqrt{5})y_1 + (2/\sqrt{5})y_2$. The equation becomes $5y_2^2 - 6\sqrt{5}y_2 + 9 = (\sqrt{5}y_2 - 3)^2 = 0$, a double line.

2. By lectures, the first step is to rotate axes by $\pi/4$. New coordinates are y_1, y_2, where $x_1 = (y_1 - y_2)/\sqrt{2}$, $x_2 = (y_1 + y_2)/\sqrt{2}$. We get the equation $(1+c)y_1^2 + (1-c)y_2^2 = 1$. This is an ellipse for $|c| < 1$, a hyperbola if $|c| > 1$ and a pair of parallel lines if $|c| = 1$.

3. The equation derived from the focus-directrix condition is that given in Q4. As $e > 0$ the coefficients of x_1^2 and x_2^2 cannot be equal, so we never get a circle. The only value of e for which the coefficients are negatives of each other is $e^2 = 2$, i.e. $e = \sqrt{2}$.

4. (a) $a = se/(1-e^2)$, $b = se/\sqrt{1-e^2}$. Taking the positive values we get $b/a = \sqrt{1-e^2}$, so $e = \sqrt{1-(b^2/a^2)}$. Hence $r = ae = \sqrt{a^2-b^2}$ and $s = b^2/\sqrt{a^2-b^2}$.

(b) The origin of the x-system has coordinates $(r,0) = (ae,0)$ in the y-system. The line $x_1 = s$ is the same as $y_1 = s + r$. Check that $r + s = a/e$.

(c) If we multiply a by -1 we get the same equation for the ellipse, but the formulae for the focus and directrix change sign. (The ellipse is symmetric about the y_2-axis.)

(d) By the focus-directrix property the distance from x to p is e times the distance to L (where L is the directrix $y_1 = a/e$). The distance to L is $|y_1 - a/e|$, so $||x-p|| = e|y_1 - a/e|$. Similarly $||x-p'|| = e|y_1 + a/e|$. Since $y_1 \leq a/e$, the sum of these is $e(a/e - y_1 + y_1 + a/e) = 2a$.

5. Choose the coordinate system so that the Parliament and the FO are the points $A = (1,0)$ and $B = (-1,0)$. Let's prove that the curve is the ellipse of which A and B are the focuses. By Q4 (d) we know that the sum of distances from any point of the ellipse $y_1^2/a^2 + y_2^2/b^2 = 1$ to the focuses is $2a$. So we take $a = 2$. The sum of distances from $(0,b)$ to $(1,0)$ and $(-1,0)$ is $2\sqrt{b^2+1}$, so we take $b = \sqrt{3}$. What we are now going to prove is that our curve is the ellipse $y_1^2/4 + y_2^2/3 = 1$. By Q4 (d) we know that for every point of this ellipse the sum of distances to $(1,0)$ and $(-1,0)$ is 4. But we still need to show that this property holds only for the points of the ellipse and for no other points! to prove this, let D be a point outside of the ellipse, and let C be the point of intersection of the line AD and the ellipse. By the triangle inequality $4 = |AC| + |BC| < |AD| + |CD| + |BC| = |AD| + |BD|$, because B, C and D are on the same line. This proves that no point D outside of the ellipse has the property $|AD| + |BD| = 4$. A similar argument shows that this equality is never satisfied for points D in the interior of the ellipse.