M1GLA Geometry and Linear Algebra, Solutions to Sheet 5

1. (a) Unique solution $(2, -1, 3, -1)$.

(b) No solution

(c) General solution $(x_1, \ldots, x_5) = (4 - a, 10 - \frac{5}{2}b - 3a, 2 + \frac{1}{2}b, b, a)$ for any $a, b \in \mathbb{R}$.

(d) General solution $(5a, -2a, a)$.

2. Augmented matrix reduces to

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2-a & -3 \\ 0 & 0 & 2-2a & b-3 \end{pmatrix}.$$

Hence last equation is $(2 - 2a)x_3 = b - 3$. So get no solutions if $a = 1, b \neq 3$; unique solution if $a \neq 1$; and infinitely many solutions if $a = 1, b = 3$.

3. The valid products are A^2, AC, B^2, CB. Their calculation is straightforward.

4. We haven't done any theory to find A^n. But if you work out A, A^2, A^3 (maybe one or two more) you will spot a pattern:

$$A^n = \frac{1}{2}\begin{pmatrix} 3^n + (-1)^n & 3^n + (-1)^{n+1} \\ 3^n + (-1)^{n+1} & 3^n + (-1)^n \end{pmatrix}.$$

You can prove this quite easily using induction.

5. Let x_1, x_2, x_3, x_4 be the number of goals scored by A, B, C, D, respectively. As each team plays the others twice, the total number of games is 12. Hence

$$\begin{aligned} (1) \Rightarrow \quad & x_1 + x_2 + x_3 + x_4 &= 36 \\ (2) \Rightarrow \quad & x_1 - 2x_2 - 2x_3 &= 0 \\ (3),(4) \Rightarrow \quad & x_1 - x_2 - 4x_3 + 4x_4 &= 0 \end{aligned}$$

Reducing to echelon form gives last equation $x_3 - (11/9)x_4 = 4$. Since the x_i are integers, it follows that x_4 is divisible by 9. If x_4 is 9 or more then x_3 is at least 15, and equations (1),(2) are clearly impossible. Hence $x_4 = 0$ and we get $x_3 = 4, x_2 = 8, x_1 = 24$.