M1GLA Geometry and Linear Algebra, Solutions to Sheet 6

1. (i) A straightforward calculation shows that for the first two matrices we have $m = 3$ and $m = 4$, respectively. The third matrix gives $A^2 = 0$, hence any higher power is the all-zero matrix, so no m such that $A^m = I_2$ exists. For the fourth matrix an easy proof by induction shows that for $m \geq 1$ we have

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}$$

Again, no $m > 0$ such that $A^m = I_2$ exists.

(ii) If A is the first matrix, then $A^4 = I_4$. The other matrix gives $A^4 = 0$.

2. (i) The column vector Ap has coordinates $\sum_{j=1}^n a_{ij}p_j \geq 0$. Their sum equals

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j = \sum_{j=1}^n \sum_{i=1}^n a_{ij}p_j = \sum_{j=1}^n (\sum_{i=1}^n a_{ij})p_j = \sum_{j=1}^n p_j = 1.$$

The first equality is due to changing the order of summation, and the third equality uses the definition of a stochastic matrix. Thus the sum of coordinates of Ap is 1. Conversely, let p be the vector with $p_i = 1$ and $p_j = 0$ for $j \neq i$. Then Ap is the column vector $(a_{1i}, a_{2i}, \ldots, a_{ni})$. All these entries must be non-negative and sum up to 1. Doing this for all i from 1 to n we get the result.

(ii) All the terms in the sum $\sum_{j=1}^n a_{ij}p_j$ are non-negative. Thus the sum is zero means that all the terms are zero. But at least one of the p_j has to be positive, and all $a_{ij} > 0$, so $\sum_{j=1}^n a_{ij}p_j > 0$.

3. The entries of AB are $\sum_{k=1}^n a_{ik}b_{kj} \geq 0$. The sum of entries in the j-th column is

$$\sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^n (\sum_{i=1}^n a_{ik})b_{kj} = \sum_{k=1}^n b_{kj} = 1,$$

using $\sum_{i=1}^n a_{ik} = 1$ and $\sum_{k=1}^n b_{kj} = 1$.

4. Consider the matrix E_{ij} whose ij-entry is 1, and all the other entries are 0. Let the entries of A be a, b, c, d. Let's write down the restrictions on a, b, c, d imposed by the formulae $A \cdot E_{ij} = E_{ij} \cdot A$ for, say $E_{1,1}$ and $E_{1,2}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

The formulae show that $b = c = 0$ and $a = d$, hence A is a scalar matrix:

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Visibly it commutes with any other matrix.

5. The same method as in the previous question shows that these are precisely the scalar matrices.

6. Let us compare both products

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Thus the matrices A are the matrices of the form

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

for real numbers a and b.