M1GLA Geometry and Linear Algebra, Solutions to Sheet 7

1.(a) Matrices are $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Last is rotation through π.

(b) Say s sends (x_1, x_2) to (y_1, y_2). Draw diagram with $||x|| = ||y|| = r$ and x making angle ϕ with x_1-axis. Then we have $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $y_1 = r\cos(2\theta - \phi)$, $y_2 = r\sin(2\theta - \phi)$. From this we get $y_1 = x_1\cos 2\theta + x_2 \sin 2\theta$, $y_2 = x_1 \sin 2\theta - x_2\cos 2\theta$, which gives the matrix required.

(c) In lectures I showed the matrix representing sr_θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$. This is the matrix of a reflection by (b). (Replace 2θ in (b) by $-\theta$.)

(d) Let L and M be the lines $x_2 = x_1 \tan\theta_1$ and $x_2 = x_1\tan\theta_2$. By (b), the matrix representing st is

$$\begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{pmatrix} \begin{pmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{pmatrix}.$$

When you multiply this you get the matrix of the rotation through $2\theta_1 - 2\theta_2$.

2. Inverse $-\frac{1}{2}\begin{pmatrix} 9 & -7 \\ -8 & 6 \end{pmatrix}$; no inverse; inverse $\begin{pmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{pmatrix}$.

3. $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})AB = I$, so $B^{-1}A^{-1}$ is the inverse of AB.

4. (i) Invertible provided $a \neq \frac{1}{2}$

(ii) If $a \neq \frac{1}{2}$, $A^{-1} = \frac{1}{1-2a}\begin{pmatrix} 1 & -2 & -1 \\ -2a & 2 & 1 \\ -a & 1 & 1-a \end{pmatrix}$.

5. A straightforward though tedious calculation. Can you think of a more conceptual proof that would generalize to any size?