M1GLA Geometry and Linear Algebra, Solutions to Sheet 8

1. (a) 0; (b) solutions $t = -2, 4$; (c) $x = 0,\ b - c,\ (a + b + c)/2$.

2. (i) Eigenvalues 1 and 3. The corresponding eigenvectors $\begin{pmatrix} a \\ -a \end{pmatrix}$ and $\begin{pmatrix} b \\ -2b \end{pmatrix}$ (any non-zero real numbers a, b). So e.g. $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ will do.

(ii) The characteristic polynomial is $-(\lambda - 1)(\lambda - 3)^2$, so eigenvalues are 1 and 3. For $\lambda = 1$, eigenvectors are scalar multiples of $(2, -1, 1)^T$. For $\lambda = 3$ eigenvectors are $(a + b, a, b)$ (any a, b). So take e.g. $P = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. (Many other P's will work of course.)

(iii) Eigenvalues 2 and -1. For $\lambda = 2$ eigenvectors are scalar multiples of $(1, 0, 1)^T$; for $\lambda = -1$ eigenvectors are scalar multiples of $(1, -3, 4)^T$. Any P such that $P^{-1}AP$ is diagonal must have eigenvectors of A as columns, and hence two of its columns must be scalar multiples of each other; but such a matrix has determinant 0 so cannot be invertible. Hence no invertible P can be found.

3. (a) If $P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ then $P^{-1}AP = D = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$.

(b) As we have seen in the lectures, $(P^{-1}AP)^n = P^{-1}A^nP$, hence $P^{-1}A^nP = D^n$, giving $A^n = PD^nP^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 8^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. This works out as $\begin{pmatrix} 2 \cdot (-1)^n - 8^n & 2 \cdot (-1)^n - 2 \cdot 8^n \\ (-1)^{n+1} + 8^n & (-1)^{n+1} + 2 \cdot 8^n \end{pmatrix}$.

(c) If $E = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, then $E^3 = D$, so as in lectures, $(PEP^{-1})^3 = PE^3P^{-1} = PDP^{-1} = A$. So take $B = PEP^{-1} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$.

(d) If $F = \begin{pmatrix} i & 0 \\ 0 & \sqrt{8} \end{pmatrix}$ then $F^2 = D$, so setting $C = PFP^{-1}$ as in (c), we obtain that $C = \begin{pmatrix} -\sqrt{8} + 2i & -2\sqrt{8} + 2i \\ \sqrt{8} - i & 2\sqrt{8} - i \end{pmatrix}$ satisfies $C^2 = A$.

(e) From Question 5, Sheet 7 we know that $|XY| = |X||Y|$. Suppose $C^2 = A$, then $|A| = |C|^2 \geq 0$, but $|A| = -8$, a contradiction.

4. Call these matrices $A_1, \ldots, A_4$. The characteristic polynomials of these matrices are

$$t^2 + t + 1, \quad t^2 + 1, \quad t^2, \quad (t - 1)^2,$$

respectively. For A_1 we have $t^2 + t + 1 = (t + \frac{1}{2} + \frac{\sqrt{3}}{2}i)(t + \frac{1}{2} - \frac{\sqrt{-3}}{2}i)$; the eigenvalues are the complex cube roots of 1; no real eigenvalues. The eigenvalues of A_2 are $\pm i$; no real eigenvalues. The only eigenvalue of A_3 is 0, a real number. The only eigenvalue of A_4 is 1, a real number.