M1GLA Geometry and Linear Algebra
Solutions Sheet 9

1*. (a) We have $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ and $(A^{-1})^T A^T = (AA^{-1})^T = I$.

(b) Yes, by (a).

(c) Note that $(A + I_n)x = 0$ for many vectors x, e.g. $x = (1, -1, 0, \ldots)$. Similarly, $(A + (1-n)I_n)x = 0$ for $x = (1, \ldots, 1)$. Hence $|A + I_n| = |A + (1-n)I_n| = 0$.

2*. (a) $t^2 + a^2 = (t - ia)(t + ia)$.

(b) $-t(t^2 + a^2 + b^2 + c^2)$; the eigenvalues are 0 and $\pm i\sqrt{a^2 + b^2 + c^2}$.

(c) $(P^{-1}AP)^T = (P^TAP)^T = P^TA^TP = -P^TAP = -P^{-1}AP$.

(d) Yes, by Q1 (a).

3*. (a) No, e.g. $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) The characteristic polynomial of $A = \begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}$ is $(t - 10)(t - 20)$. The rotation matrix formed by orthogonal eigenvectors of length 1 is $A = \begin{pmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{pmatrix}$. The standard form is $10y_1^2 + 20y_2^2 = a$, which is an ellipse for $a > 0$, a single point if $a = 0$, and the empty set if $a < 0$.

4*. (a) Any symmetric matrix with real entries is diagonalizable, that is, can be written as $P^{-1}DP$, where D is diagonal with ± 1 on the main diagonal. We have $D^{-1} = D = D^T$. Now $(P^{-1}DP)^{-1} = P^{-1}DP = P^TD^TP = (P^{-1}DP)^T$.

(b) $A^3 = I_3$, as can be checked by a straightforward calculation. A more intelligent proof is by observing that A can be diagonalized. Indeed, the eigenvalues of A are $1, \zeta, \zeta^2$, and these are distinct. Thus we can write $A = PDP^{-1}$ for some invertible matrix P with complex entries, and the diagonal matrix D with diagonal entries $1, \zeta, \zeta^2$. Therefore, $A^3 = PD^3P^{-1} = PP^{-1} = I_3$.

5. (a) is obvious.

(b) $\lambda\bar{v}.v = (\bar{v})^TAv = (A^T\bar{v})^Tv = \bar{\lambda}\bar{v}.v$, using (a), hence the result. (c) is similar.

(d) Note that $A^{-1}v = \lambda^{-1}v$, then proceed as in (b).