# A very brief introduction to étale homotopy*

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The task of these notes is to supply the reader who has little or no experience of simplicial topology with a phrase-book on étale homotopy, enabling them to proceed directly to [5] and [10]. This text contains no proofs, for which we refer to the foundational book by Artin and Mazur [1] in the hope that our modest introduction will make it more accessible. This is only a rough guide and is no substitute for a rigorous and detailed exposition of simplicial homotopy for which we recommend [8] and [4].

Let $X$ be a Noetherian scheme which is locally unibranch (this means that the integral closure of every local ring of $X$ is again a local ring), e.g., a Noetherian normal scheme (all local rings are integrally closed). All smooth schemes over a field fall into this category. The aim of the Artin–Mazur theory is to attach to $X$ its étale homotopy type $\acute{E}t(X)$. This is an object of a certain category pro $-\mathcal{H}$, the pro-category of the homotopy category of CW-complexes. The aim of these notes is to explain this construction.

# 1 Simplicial objects

## 1.1 Simplicial objects and sisets

The ordinal category $\Delta$ is the category whose objects are finite ordered sets $[n] = \{0, \ldots, n\}$, one for each non-negative integer, where the morphisms are the order-preserving maps $[n] \to [m]$. It is easy to see that all these maps are compositions of the *face* maps $\delta_i : [n] \to [n+1]$ (the image misses $i$) and the *degeneracy* maps $\sigma_i : [n+1] \to [n]$ (hitting $i$ twice).

A *simplicial object* with values in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. Simplicial objects in $\mathcal{C}$ form a category $\mathcal{SC}$, where the morphisms are the natural transformations of such functors.

For example, a simplicial set (or a *siset*) is a simplicial object $S = (S_n)$ with values in $\mathcal{S}ets$, the category of sets. Let $\mathcal{SS}ets$ be the category of sisets. The elements of

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*The notes of Tomer Schlank's lectures at the study group organised by Ambrus Pál at Imperial College London in the Autumn of 2010, taken by Alexei Skorobogatov. We thank Yonatan Harpaz for useful comments.

$S_n$ are called $n$-simplices of $S$. A simplex $x$ is non-degenerate if it is not of the form $\sigma_i(y)$ for some simplex $y \in S$.

We denote by $\Delta[n]$ the siset given by the contravariant functor $\Delta \to \mathcal{S}ets$ sending $[m]$ to $\mathrm{Hom}_\Delta([m],[n])$. For example, $\Delta[0]$ has the one-element set in every degree $m$, that is, the zero function $[m] \to [0]$. Next, $\Delta[1]_m$ can be identified with the set of non-decreasing sequences of 0 and 1 of length $m+1$. The degeneracy operators repeat the $i$-th coordinate; the face operators erase the $i$-th coordinate. The siset $\Delta[1]$ has only three non-degenerate simplices: the constant functions $[0] \to 0$ and $[0] \to 1$ in degree 0, and the identity function $[1] \to [1]$ in degree 1. In general, $\Delta[n]_m$ is the set of increasing sequences of $0, 1, \ldots, n$ of length $m+1$, so that the identity $[n] \to [n]$ is the unique non-degenerate $n$-simplex of $\Delta[n]$. For any $n$-simplex of a siset $S$ there is a unique map $\Delta[n] \to S$ that sends the identity function $[n] \to [n]$ to this simplex, hence $S_n = \mathrm{Hom}_{\mathcal{SS}ets}(\Delta[n], S)$.

The product of two sisets $R \times S$ is defined levelwise: $(R \times S)_n = R_n \times S_n$ with face and degeneracy operators acting simultaneously on both factors.

The maps of sisets $f : R \to S$ and $g : R \to S$ are strictly homotopy equivalent if there is a map of sisets $R \times \Delta[1] \to S$ whose restrictions to $R \times (0)$ and $R \times (1)$ give $f$ and $g$, respectively. Two maps of sisets are homotopy equivalent if they can be connected by a chain of strict homotopies.

If $S$ is a siset and $A$ is an abelian group, then the *simplicial cohomology groups* $\mathrm{H}^n(S, A)$ are defined as the cohomology groups of the complex

$$\ldots \to \mathrm{Maps}(S_1, A) \to \mathrm{Maps}(S_0, A) \to 0,$$

where the differentials are alternating sums of the maps induced by the face maps.

## 1.2 Topological realisation

Let $Top$ be the category of topological spaces, where the morphisms are continuous maps. Recall that a continuous map of topological spaces is a *weak equivalence* if it induces isomorphisms on all the homotopy groups.

Consider the standard $n$-dimensional simplex:

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0, \sum x_i = 1\}.$$

Any siset $S$ has the *topological realisation* $|S|$ which is a topological space defined as the quotient of the disjoint union of $S_n \times \Delta^n$ by the equivalence relation generated by the following relations: for any order preserving map $\alpha : [m] \to [n]$ we identify $(\alpha^*(x), t)$ with $(x, \alpha(t))$ (it is enough to do this for the face maps and the degeneracy maps). The natural map $|\Delta[n]| \to \Delta^n$ is a homeomorphism. The topological realisation $|S|$ of a siset $S$ can be given the structure of a CW-complex: the $n$-cells are all non-degenerate $n$-simplices of $S$, with the face maps as the gluing maps, see [8, Thm. 14.1].

Any map of sisets induces a continuous map of topological realisations, hence the topological realisation is a functor $\mathcal{S}Sets \to Top$. Homotopic maps of sisets give rise to topologically homotopic maps. One can define the homotopy groups of a pointed siset directly, see [4, I.7], or, equivalently, as the homotopy groups of its topological realisation. The topological realisation commutes with products: we have $|R \times S| = |R| \times |S|$ for any sisets $R$ and $S$ such that $|R| \times |S|$ is a CW-complex (e.g. when $R$ and $S$ are both countable). See [8], Thm. 14.3 and Remark 14.4.

The *singular functor* $Top \to \mathcal{S}Sets$ attaches to a topological space $T$ the siset $ST$, where $(ST)_n$ is the set of continuous maps $\Delta_n \to T$. It is a crucial property that the singular functor is right adjoint to the functor of topological realisation.

Let $\mathcal{H}$ be the homotopy category of CW-complexes. If $X$ and $Y$ are CW-complexes, then the set of morphisms from $X$ to $Y$ in $\mathcal{H}$ is the set of homotopy classes of maps $X \to Y$. It is denoted by $[X, Y]$.

Let $\mathcal{H}_0$ be the pointed homotopy category of connected pointed CW-complexes (the homotopy is assumed to preserve the base point).

## 1.3 Simplicial mapping space

In the category of sisets there is a natural notion of internal Hom.

**Definition 1.1** *Let $A$ and $B$ be sisets. The internal mapping space $\underline{\mathrm{Maps}}(A, B)$ is an object of $\mathcal{S}Sets$ defined by*

$$\underline{\mathrm{Maps}}(A, B)_n := \mathrm{Hom}_{\mathcal{S}Sets}(A \times \Delta[n], B),$$

*with the natural face and degeneracy maps.*

There is a natural map

$$\phi : |\underline{\mathrm{Maps}}(A, B)| \to \mathrm{Maps}_{Top}(|A|, |B|).$$

However, sisets are too rigid in the sense that some maps of their realisations cannot be represented up to homotopy by maps of sisets, so that $\phi$ is not always a weak equivalence. For example, let $C$ be the siset obtained by identifying the two 0-simplices of $\Delta[1]$. It is clear that $|C|$ is homotopy equivalent to $S^1$, however one can check that the only maps $C \to C$ are the constant map and the identity. Thus all the surjective maps $S^1 \to S^1$ of degrees more than 1 are not in the image of $|\underline{\mathrm{Maps}}(C, C)| \to \mathrm{Maps}_{Top}(|C|, |C|)$.

Kan noticed that $\phi$ is a weak equivalence if $B$ (but not necessarily $A$) belongs to a special class of sisets that are now called Kan sisets.

## 1.4 Kan sisets and Kan fibrations

Let $m \in \{0, \ldots, n\}$. The *horn* $\Lambda^m[n]$ is the smallest sub-siset of $\Delta[n]$ containing all the non-degenerate $(n-1)$-simplices of which $m$ is an element (that is, all except one). A map of sisets $p : X \to B$ is a *Kan fibration* if for any $n \geq 1$ and any commutative diagram

$$\begin{tikzcd} \Lambda^m[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a dotted map of sisets such that the two resulting triangles are commutative ("every horn has a filler"). If $(B, b)$ is a pointed siset, then $p^{-1}(b)$ is called the fibre of $f$. A siset $Y$ is called *fibrant* or a *Kan siset* if the map sending $Y$ to a point is a Kan fibration. For example, if $T$ is a topological space, then $ST$ is a Kan siset. Note, however, that $\Delta[n]$ is a not a Kan siset for $n \geq 1$ (exercise).

Recall that a continuous map of topological spaces $f : V \to U$ is a *Serre fibration* if the dotted arrow exists in every commutative diagram of commutative maps

$$\begin{tikzcd} |\Lambda^m[n]| \arrow[r] \arrow[d, hook] & V \arrow[d, "f"] \\ |\Delta[n]| \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

turning it into two commutative triangles. Kan fibrations of simplicial sets behave similarly to Serre fibrations of topological spaces. For example, Kan fibrations have the homotopy lifting property: if $E \to B$ is a Kan fibration and $X$ is any siset, then for every diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \times \Delta[1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a dotted map of sisets such that the two resulting triangles are commutative. The adjointness of $S$ and $|\ |$ implies that $f : V \to U$ is a Serre fibration if and only if $f : S(V) \to S(U)$ is a Kan fibration. Quillen proved that if $X \to B$ is a Kan fibration, then $|X| \to |B|$ is a Serre fibration, see [4, Thm. I.10.10].

As mentioned above, if $A$ is a siset and $B$ is a Kan siset, then the natural map $\phi$ from the previous section is a weak equivalence[1]. This fact makes Kan simplicial sets a good model for topological spaces. Indeed, the homotopy category of Kan sisets (where the morphisms are the simplicial homotopy classes of maps) is equivalent to the homotopy category of CW-complexes, via the topological realisation functor and the singular functor. Thus in applications is it possible to use constructions from either category.

For an arbitrary siset $A$ the adjunction between realisation and the singular functors gives rise to a natural map

$$A \to S|A|.$$

[1]This is a consequence of the fact that sisets and topological spaces are Quillen equivalent as simplicial model categories, and the Kan sisets are fibrant in the model category of sisets. See [6] for more on this subject.

This map is a weak equivalence, and $S|A|$ is a Kan siset (cf. [4], the proof of Thm. 11.4). Therefore, every siset can be functorially replaced with a weakly equivalent Kan siset. It is common to use for this purpose a different functor with a better combinatorial description, namely the Kan replacement functor $\mathrm{Ex}^\infty$, see [4, Thm. III.4.8].

# 2 Pro-categories

See the appendix to [1] for more details.

**Definition 2.1** *A category is cofiltering if it satisfies the following conditions:*

(1) *for any objects $A$ and $B$ there is an object $C$ with morphisms to both $A$ and $B$,*

(2) *if there are morphisms* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *, then there is a morphism $h : C \to A$ such that $fh = gh$.*

In other words, there is a diagram $C \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$ .

**Examples** (1) The category whose objects are natural numbers, and the morphisms are $n \to m$ whenever $n \geq m$.

(2) The category whose objects are positive natural numbers, and the morphisms are $n \to m$ whenever $m|n$.

(3) Connected pointed étale coverings of a pointed scheme.

**Definition 2.2** *Let $\mathcal{C}$ be a category. The objects of the* **pro-category** $\mathrm{pro}-\mathcal{C}$ *are functors $F : \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ is cofiltering. The morphisms are defined as follows:*

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{C}}(\{C_i\}_{i\in\mathcal{I}}, \{D_j\}_{j\in\mathcal{J}}) = \varprojlim_{j\in\mathcal{J}} \varinjlim_{i\in\mathcal{I}} \mathrm{Hom}(C_i, D_j).$$

This definition of morphisms between pro-objects should be familiar: homomorphisms of pro-finite groups are defined in the same way.

Define $\mathrm{pro}-\mathcal{H}$ as the pro-category of the homotopy category of CW-complexes $\mathcal{H}$. For the objects of $\mathrm{pro}-\mathcal{H}$ one defines the analogues of homology groups, but note that these are defined as pro-groups, not groups. Let $\mathrm{pro}-\mathcal{H}_0$ be the pro-category of the pointed homotopy category of connected pointed CW-complexes $\mathcal{H}_0$. For the objects of $\mathrm{pro}-\mathcal{H}$ one defines the homotopy pro-groups.

# 3 Coverings and hypercoverings

## 3.1 Coverings of CW-complexes and étale coverings of schemes

Let $X$ be a CW-complex. An open covering $X = \cup_{\alpha \in J} U_\alpha$ is called *excellent* if the intersection of any number of open sets $U_\alpha$ is contractible or empty. The homotopy type of $X$ can be recovered from the siset attached to such a covering. Indeed, let $\mathbf{U}_n$ be the set of functions $f : [n] \to J$ such that $\cap_{i=1}^n U_{f(i)} \neq \emptyset$. The face and degeneracy maps in $\mathbf{U}_\bullet$ are defined in the obvious way. Then the topological realisation $|\mathbf{U}_\bullet|$ is weakly equivalent to $X$. This means that $X$ and $|\mathbf{U}_\bullet|$ can be connected by a zigzag of morphisms that induce isomorphisms on all homotopy groups.

We can think of a covering $X = \cup_{\alpha \in J} U_\alpha$ as a map $\mathcal{U} \to X$, where $\mathcal{U}$ is the disjoint union of open sets $U_\alpha$. Note that $U_\alpha \cap U_\beta = U_\alpha \times_X U_\beta$, so that the fibred product $\mathcal{U} \times_X \mathcal{U}$ is just the disjoint union of pairwise intersections $U_\alpha \cap U_\beta$, the fibred product $\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}$ is the disjoint union of triple intersections, and so on.

**Definition 3.1** *Let $\mathcal{U} \to X$ be a covering of a topological space $X$. The Čech nerve $\pi_0(\mathcal{U})$ of $\mathcal{U} \to X$ is the siset such that*

$$\pi_0(\mathcal{U})_n = \pi_0(\mathcal{U} \times_X \ldots \times_X \mathcal{U}) \qquad (n+1 \text{ times}).$$

*The face maps in $\pi_0(\mathcal{U})$ are obvious projections, and the degeneracy maps are various diagonal embeddings.*

If we allow all the $n$-fold intersections to be disjoint unions of contractible sets, we arrive at the notion of a *good* covering. If $\mathcal{U} \to X$ is a good covering, then the topological realisation $|\pi_0(\mathcal{U})|$ of its Čech nerve is weakly equivalent to $X$.

Representing a covering by a morphism $\mathcal{U} \to X$ is convenient because this carries over to étale coverings. One would like to apply this construction to a Grothendieck topology on a scheme $X$, for example, to the small étale site $X_{\text{ét}}$ of $X$. The obvious problem is that open étale "subsets" are rarely contractible. However, it is clear that to recover the space we started with, the connected components must be contractible. Instead, we can think of the Čech nerve of an étale covering $\mathcal{U} \to X$ as an "approximation" to the homotopy type of the scheme $X$. This approximation becomes better when we take finer and finer coverings. On passing to a limit in $\mathcal{U}$ this system of approximations computes the Čech cohomology of $X$. Pro-objects in étale homotopy theory are used exactly in order to formalise this notion of "a hierarchic system of approximations". Naively, one might want to define the étale homotopy type as the pro-space $\acute{E}t : Cov(X) \to Top$, where $Cov(X)$ is the category of étale coverings of $X$, and $\acute{E}t(\mathcal{U}) = |\pi_0(\mathcal{U})|$ is the topological realisation of the Čech nerve of $\mathcal{U}$. There are two problems with this definition. The first problem is that the system of étale coverings does not always compute the correct cohomology groups. The second problem is that the category of étale coverings $Cov(X)$ is not cofiltering (cf. [9], III, Remark 2.2 (a)).

To understand the first problem better, we note that for an étale covering $\mathcal{U} \to X$ the cohomology groups $\mathrm{H}^n(|\pi_0(\mathcal{U})|, A)$ coincide with the Čech étale cohomology groups $\check{\mathrm{H}}^n(\mathcal{U}, A)$ (by definition, see [9], III.2). Passing to the limit at the level of cohomology groups is well defined (see [9], *loc. cit.*). Thus we obtain the groups

$$\check{\mathrm{H}}^n_{\text{ét}}(X, A) = \varinjlim \check{\mathrm{H}}^n(\mathcal{U}, A).$$

Unfortunately, $\check{\mathrm{H}}^n_{\text{ét}}(X, A)$ is not always equal to the étale cohomology group $\mathrm{H}^n_{\text{ét}}(X, A)$. For a quasicompact scheme $X$ the canonical morphism $\check{\mathrm{H}}^n_{\text{ét}}(X, A) \to \mathrm{H}^n_{\text{ét}}(X, A)$ is an isomorphism when any finite subscheme of $X$ is contained in an affine subset, e.g., $X$ is quasi-projective over an affine scheme (for example, $X$ is quasi-projective over a field). This is Artin's theorem, see [9, III.2.17].

A theorem of Verdier points at a way to remedy this situation by replacing coverings by a more general notion of hypercoverings. To define hypercoverings we first need to introduce the notions of skeleton and coskeleton which we do in the next section.

Unfortunately, even replacing the category $Cov(X)$ by the category of hypercoverings $Hyp(X)$ is not enough to solve the second problem. The reason is that $Hyp(X)$ is not cofiltering, either. To circumvent this problem one can work with the notion of the homotopy category of hypercoverings $HC(X)$ which is cofiltering. But passing to the homotopy classes of hypercoverings comes with a price: we obtain an object in $Pro - \mathcal{H}$ rather than in $Pro - Top$. This suffices for our needs, but in some cases one would like to get the actual "étale topological type". This can be done; we refer the interested reader to Friedlander's book [3] and to the recent paper [2].

## 3.2 Skeleton and coskeleton

Let $\mathcal{C}$ be a category closed under finite limits and colimits (e.g. the category of sets $\mathcal{S}ets$, the category of pointed sets, or the étale site of a scheme). For $n \geq 0$ one defines functors $\mathrm{sk}_n$ and $\mathrm{cosk}_n$ from $\mathcal{SC}$ to itself, as follows. Let $\Delta/n$ be the ordinal category $\Delta$ truncated at level $n$, i.e. the full subcategory of $\Delta$ whose objects are $[m]$ for $m \leq n$. Let $\mathcal{S}_n\mathcal{C}$ be the category of functors $\Delta/n \to \mathcal{C}$. Since $\Delta/n$ is a finite category and $\mathcal{C}$ is closed under finite colimits, the obvious truncation functor $\tau_n : \mathcal{SC} \to \mathcal{S}_n\mathcal{C}$ has a left adjoint (the left Kan extension, cf. [7, X.4]). The composition of the truncation functor and its left adjoint

$$\mathrm{sk}_n : \mathcal{SC} \to \mathcal{S}_n\mathcal{C} \to \mathcal{SC}$$

is called the *skeleton* functor. For example, if $\mathcal{C} = \mathcal{S}ets$, then $\mathrm{sk}_n(X)$ is the simplicial subset of $X$ that agrees with $X$ up to the level $n$, and which has no non-degenerate simplices in dimensions greater than $n$.

Since $\Delta/n$ is a finite category and $\mathcal{C}$ is closed under finite limits, the functor $\tau_n : \mathcal{SC} \to \mathcal{S}_n\mathcal{C}$ also has a right adjoint (the right Kan extension, cf. [7, X.3]). The composed functor

$$\mathrm{cosk}_n : \mathcal{SC} \to \mathcal{S}_n\mathcal{C} \to \mathcal{SC}$$

is called the *coskeleton* functor. By definition we have

$$\mathrm{Hom}_{\mathcal{SC}}(\mathrm{sk}_n(A), B) = \mathrm{Hom}_{\mathcal{S}_n\mathcal{C}}(\tau_n(A), \tau_n(B)) = \mathrm{Hom}_{\mathcal{SC}}(A, \mathrm{cosk}_n(B)),$$

and taking $A = B$ we obtain the canonical morphisms

$$\mathrm{sk}_n(A) \to A \quad \text{and} \quad A \to \mathrm{cosk}_n(A).$$

Now let $\mathcal{C}$ be the category of sets or pointed sets. If $A$ is a (pointed) siset, then it is clear that $\mathrm{cosk}_n(A)$ is the siset such that

$$\mathrm{cosk}_n(A)_m = \mathrm{Hom}_{\mathcal{SS}ets}(\Delta[m], \mathrm{cosk}_n(A)) = \mathrm{Hom}_{\mathcal{SS}ets}(\mathrm{sk}_n(\Delta[m]), A).$$

In particular, $\mathrm{cosk}_0(A)$ is the siset such that

$$\mathrm{cosk}_0(A)_m = \mathrm{Hom}_{\mathcal{S}ets}(\mathrm{sk}_0(\Delta[m]), A) = A_0^{m+1}.$$

Also, $\mathrm{cosk}_n(\Delta[r]) = \Delta[r]$ if $n \geq r$.

The coskeleton functor preserves Kan sisets (but not Kan fibrations[2]). If $X$ is an object of $\mathcal{H}_0$, then the coskeleton $\mathrm{cosk}_n(X)$ is characterised by the following universal property: $\pi_m(\mathrm{cosk}_n(X)) = 0$ for $m \geq n$, and the canonical map $X \to \mathrm{cosk}_n(X)$ is universal in the homotopy category among the maps to objects with vanishing $\pi_m$ for $m \geq n$, cf. [1], (2.4). For $m < n$ the map $X \to \mathrm{cosk}_n(X)$ induces an isomorphism $\pi_m(X) \xrightarrow{\sim} \pi_m(\mathrm{cosk}_n(X))$. In other words,

$$\ldots \to \mathrm{cosk}_{n+1}(X) \to \mathrm{cosk}_n(X) \to \ldots \to \mathrm{cosk}_0(X)$$

is a Postnikov tower of $X$.

Next, the homotopy fibre of $\mathrm{cosk}_{n+1}(X) \to \mathrm{cosk}_n(X)$ is the Eilenberg–MacLane space $K(\pi_n(X), n)$. Recall that for a group $G$ a connected CW-complex $X$ is called an Eilenberg–MacLane space $K(G, n)$ if the only non-trivial homotopy group of $X$ is $\pi_n(X) = G$; all such CW-complexes are homotopy equivalent.

An important fact in homotopy theory is Whitehead's theorem that states that a map of CW-complexes is a homotopy equivalence if and only if it induces isomorphism on all homotopy groups. Whitehead's theorem allows us to study homotopy types "one homotopy group at a time" which makes it an essential tool in homotopy theory. However the analog of Whitehead's theorem is false for pro-CW-complexes. To remedy this situation Artin and Mazur [1, §3] presents the following construction.

For an object $(X_\bullet) = (X_i)$ of pro $-\,\mathcal{H}_0$ the universal property of coskeleton allows us to define the following object of pro $-\,\mathcal{H}_0$:

$$X^\natural = \{\mathrm{cosk}_n(X_i)\}.$$

[2]As it is incorrectly stated on page 8 of [1].

It is indexed by pairs $(n, i)$ with the obvious canonical maps $\mathrm{cosk}_m(X_i) \to \mathrm{cosk}_n(X_j)$, $m \geq n$, where $i \to j$ is a morphism in the indexing category $\mathcal{I}$ of $X$. The canonical map $X \to X^\natural$ is a weak homotopy equivalence, but not necessarily an isomorphism in pro $- \mathcal{H}_0$. Indeed, if $X$ is a CW-complex, then

$$[X^\natural, X] = \varinjlim [\mathrm{cosk}_n(X), X],$$

so for the canonical map $X \to X^\natural$ to be invertible in $\mathcal{H}_0$, $X$ must be *bounded*, i.e., $\pi_n(X) = 0$ for large $n$. In such a case $X$ is homotopy equivalent to $\mathrm{cosk}_n(X)$ for some $n$.

By functoriality, any map $f : X \to Y$ in pro $- \mathcal{H}_0$ induces a map $X^\natural \to Y^\natural$. The latter is an isomorphism if and only if $f$ induces isomorphisms of all coskeletons of $X$ and $Y$, and, equivalently, of all homotopy pro-groups of $X$ and $Y$ (see [1, Cor. 4.4]).

The functor $X \to X^\natural$ can also be defined in pro $- \mathcal{H}$. Informally speaking, a base point is not needed to "Postnikov-filter" the homotopy information by dimension, but is required to define the "associated graded filtration" in terms of groups.

## 3.3 Hypercoverings

Let $X$ be a scheme, and let $X_{\text{ét}}$ be the small étale site of $X$. This is the category of all schemes $Y$ étale over $X$; the coverings in $X_{\text{ét}}$ are surjective families of étale morphisms. To a covering $Y \to X$ we can associate the simplicial étale $X$-scheme $(Y_\bullet)$, where

$$Y_n = Y \times_X \ldots \times_X Y \quad (n \text{ times}).$$

The simplicial scheme $(Y_\bullet)$ is an example of a hypercovering.

**Definition 3.2** *A simplicial étale $X$-scheme $\mathcal{U}_\bullet$ is called a* **hypercovering** *if*

(1) $\mathcal{U}_0 \to X$ *is a covering;*

(2) *for every $n$ the canonical morphism $\mathcal{U}_{n+1} \to \mathrm{cosk}_n(\mathcal{U}_\bullet)_{n+1}$ is a covering.*

Note that $\mathcal{U}_0 \to X$ is a covering, by (1), and $\mathrm{cosk}_0(\mathcal{U}_\bullet)$ is the corresponding simplicial étale $X$-scheme

$$\mathrm{cosk}_0(\mathcal{U}_\bullet)_n = \mathcal{U}_0 \times_X \ldots \times_X \mathcal{U}_0.$$

The definition of a hypercovering implies that $\mathcal{U}_1 \to \mathcal{U}_0 \times_X \mathcal{U}_0$ is a covering, so it refines the notion of covering by allowing greater freedom at every level. This very general categorical construction is due to Verdier (SGA 4, Exp. V). It can be used with any site on $X$, see [1, Ch. 8].

A very important example is when $X$ is a point and the site is the category $\mathcal{S}ets$, where a covering is just a surjective family of maps. Then a hypercovering is the same thing as a *contractible* Kan siset, see Enlightenment 8.5 (a) in [1].

## 3.4 Homotopy category of hypercoverings

Assume that $\mathcal{C}$ is a category with finite direct sums. If $K$ is an object of $\mathcal{C}$, and $S$ is a finite set, then $K \otimes S$ denotes the direct sum of copies of $K$ indexed by the elements of $S$. Now let $(K_\bullet)$ be a simplicial object with values in $\mathcal{C}$. Define the simplicial object $(K_\bullet) \otimes \Delta[1]$ in $\mathcal{SC}$ by the formula

$$((K_\bullet) \otimes \Delta[1])_n = K_n \otimes \Delta[1]_n,$$

with the simultaneous action of face and degeneracy operators on both factors. There are two obvious inclusions $\Delta[0] \to \Delta[1]$, indexed by 0 and 1. Let $e_0$ and $e_1$ be the corresponding inclusions $(K_\bullet) \to (K_\bullet) \otimes \Delta[1]$.

**Definition 3.3** *The maps $f_0 : (K_\bullet) \to (L_\bullet)$ and $f_1 : (K_\bullet) \to (L_\bullet)$ are strictly homotopic if there is a map $(K_\bullet) \otimes \Delta[1] \to (L_\bullet)$ such that $f_0 = fe_0$ and $f_1 = fe_1$. Two maps are homotopic if they are related by a chain of strict homotopies.*

We apply this to the case when $\mathcal{C}$ is the small étale site $X_{\text{ét}}$.

**Definition 3.4** *The homotopy category of hypercoverings $HC(X_{\text{ét}})$ is the category whose objects are étale hypercoverings of $X$, and whose maps are homotopy classes of morphisms of simplicial étale $X$-schemes.*

An important result is that $HC(X_{\text{ét}})$ is a cofiltering category [1, Cor. 8.13 (i)]. Passing to the limit over $HC(X_{\text{ét}})$ one establishes a canonical isomorphism [1, Thm. 8.16]

$$\mathrm{H}^n_{\text{ét}}(X, F) = \varinjlim \mathrm{H}^n(\mathcal{U}_\bullet, A).$$

We will also need the pointed versions of the above constructions: the pointed étale site on $X$ (with the choice of a geometric point on every étale $X$-scheme), pointed sisets, pointed hypercoverings, homotopy classes of pointed morphism (the strict homotopies are assumed to preserve the base point).

## 3.5 Étale homotopy type

Now we are ready to implement the strategy outlined earlier.

Let $X$ be a locally Noetherian scheme. Then every scheme $Y$ étale over $X$ is a finite disjoint union of connected schemes. Write $\pi_0(Y)$ for the set of connected components of $Y$. Let $\pi_0(\mathcal{U}_\bullet)$ be the siset obtained by applying the functor $\pi_0$ to a simplicial étale hypercovering $\mathcal{U}_\bullet$. Since $HC(X_{\text{ét}})$ is cofiltering we can consider the pro-object

$$\pi_0(X_{\text{ét}}) = \{\pi_0(\mathcal{U}_\bullet)\},$$

which is an object in the pro-category of the homotopy category of sisets. Applying the topological realisation functor we obtain the *étale homotopy type* $\acute{E}t(X)$ of $X$ as an object in $\mathrm{pro} - \mathcal{H}$.

When a base point is fixed, one can define the homology and homotopy pro-groups of $\acute{E}t(X)$. The situation with cohomology is much better because cohomology is contravariant, the direct limit is an exact functor and the direct limit of abelian groups is also an abelian group. For an abelian group $A$ this leads to the definition of the cohomology groups $\mathrm{H}^n(\acute{E}t(X), A)$. Using the previous theory one obtains a canonical isomorphism

$$\mathrm{H}^n(\acute{E}t(X), A) = \mathrm{H}^n_{\text{ét}}(X, A),$$

see [1, Cor. 9.3].

If $X$ is equipped with a geometric base point, then we can consider the pointed étale site of $X$ and define $\acute{E}t(X)$ as an object in $\mathrm{pro} - \mathcal{H}_0$. A base point allows us to define homotopy pro-groups. By [1, Cor. 10.7] for every pointed scheme we have

$$\pi_1(\acute{E}t(X)) = \pi_1(X_{\text{ét}}).$$

This makes it possible to define higher étale homotopy pro-groups for all $n \geq 0$

$$\pi_n(X_{\text{ét}}) := \pi_n(\acute{E}t(X)).$$

## 3.6 Profinite completion and the comparison theorem

Let $X$ be a pointed connected geometrically unibranch scheme over $\mathbb{C}$. Let us consider $X(\mathbb{C})$ as a topological space with the classical topology on $\mathbb{C}$. By the Riemann existence theorem we have

$$\pi_1(X_{\text{ét}}) = \widehat{\pi_1(X(\mathbb{C}))},$$

where $\widehat{G}$ is the profinite completion of the group $G$, see [9, §5]. It is natural to ask if for $n > 1$ there is an isomorphism between $\pi_n(X_{\text{ét}})$ and the profinite completion of $\pi_n(X(\mathbb{C}))$. In general this is not the case. Here one could recall a guiding principle of homotopy theory that says that functors should always be applied to homotopy types rather than to homotopy groups.

Let $\mathcal{H}_0^{\text{fin}}$ be the full subcategory of $\mathcal{H}_0$ consisting of pointed connected CW-complexes all of whose homotopy groups are finite. The following result is [1, Thm. 3.4].

**Theorem 3.5** *For any $X$ in* $\mathrm{pro} - \mathcal{H}_0$ *there are an object $\widehat{X}$ of* $\mathrm{pro} - \mathcal{H}_0^{\text{fin}}$ *and a map $X \to \widehat{X}$, which are universal with respect to maps from $X$ to objects of* $\mathrm{pro} - \mathcal{H}_0^{\text{fin}}$.

Let us call $\widehat{X}$ the *profinite completion* of $X$. Associating to $X$ its profinite completion defines a left adjoint functor to the inclusion of $\mathrm{pro} - \mathcal{H}_0^{\text{fin}}$ into $\mathrm{pro} - \mathcal{H}_0$.

Now we have the following generalisation of the isomorphism $\pi_1(X_{\text{ét}}) = \widehat{\pi_1(X(\mathbb{C}))}$, see [1], Thm. 12.9 and Cor. 12.10.

**Theorem 3.6** *Let $X$ be a pointed connected geometrically unibranch scheme over $\mathbb{C}$. Let $Cl(X)$ be the object of $\mathcal{H}_0$ given by the homotopy type of the topological space $X(\mathbb{C})$. Consider $\mathcal{H}_0$ as a subcategory of* $\mathrm{pro} - \mathcal{H}_0$*. There is a natural map*

$$Cl(X) \to \acute{E}t(X)$$

*which makes $\acute{E}t(X)$ the profinite completion of $Cl(X)$.*

Note that in the light of this theorem $\acute{E}t(X)$ can be computed from the homotopy type of $X(\mathbb{C})$, which makes $\acute{E}t(X)$ a tractable object.

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