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A FINITENESS THEOREM FOR THE BRAUER GROUP OF ABELIAN VARIETIES AND $K3$ SURFACES

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Abstract

Let k be a field finitely generated over the field of rational numbers, and $\mathrm{Br}(k)$ the Brauer group of k. For an algebraic variety X over k we consider the cohomological Brauer–Grothendieck group $\mathrm{Br}(X)$. We prove that the quotient of $\mathrm{Br}(X)$ by the image of $\mathrm{Br}(k)$ is finite if X is a $K3$ surface. When X is an abelian variety over k, and $\overline{X}$ is the variety over an algebraic closure $\overline{k}$ of k obtained from X by the extension of the ground field, we prove that the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\overline{X})$ is finite.

1. Introduction

Let X be a geometrically integral smooth projective variety over a field k. The Tate conjecture for divisors on X [30, 32, 34] is well known to be closely related to the finiteness properties of the cohomological Brauer–Grothendieck group $\mathrm{Br}\,(X) = \mathrm{H}^2_{ét}(X, \mathbf{G}_m)$. This fact was first discovered in the case of a finite field k by Artin and Tate ([31], see also Milne [18]) who studied the Brauer group of a surface. In particular, the order of $\mathrm{Br}\,(X)$ appears in the formula for the leading term of the zeta function of X. A stronger variant of the Tate conjecture for divisors concerns the order of the pole of the zeta function of X at $s = 1$; see [30, (12) on p. 101]. It implies the finiteness of the prime-to-p component of $\mathrm{Br}\,(X)$, where X is a variety of arbitrary dimension, and k is a finite field of characteristic p, as proved in [40, Sect. 2.1.2 and Remark 2.3.11].

Since Manin observed that the Brauer group of a variety over a number field provides an obstruction to the Hasse principle [17], the Brauer groups of varieties over fields of characteristic 0 have been intensively studied. Most of the existing literature is devoted to the so-called algebraic part $\mathrm{Br}_1(X)$ of $\mathrm{Br}\,(X)$, defined as the kernel of the natural map $\mathrm{Br}\,(X) \to \mathrm{Br}\,(\overline{X})$, where $\overline{X} = X \times_k \overline{k}$, and $\overline{k}$ is a separable closure of k. Meanwhile, if k is a number

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field, the classes surviving in $\mathrm{Br}(\overline{X})$ can produce a non-trivial obstruction to the Hasse principle and weak approximation (see [12] and [36] for explicit examples). Therefore, such arithmetic applications require the knowledge of the whole Brauer group $\mathrm{Br}(X)$.

To state and discuss our results we introduce some notation and conventions. In this paper the expression 'almost all' means 'all but finitely many'. If B is an abelian group, we denote by B_{tors} the torsion subgroup of B, and write $B/_{\mathrm{tors}} := B/B_{\mathrm{tors}}$. For a prime ℓ let $B(\ell)$ be the subgroup of B_{tors} consisting of the elements whose order is a power of ℓ, and $B(\mathrm{non}-\ell)$ the subgroup of B_{tors} consisting of the elements whose order is *not* divisible by ℓ. If m is a positive integer, we write B_m for the kernel of the multiplication by m in B.

Let $\mathrm{Br}_0(X)$ be the image of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$. Recall that both $\mathrm{Br}(X)$ and $\mathrm{Br}(\overline{X})$ are torsion abelian groups whenever X is smooth; see [11, II, Prop. 1.4]. There are at least three reasons why the Brauer group $\mathrm{Br}(X)$ can be infinite: $\mathrm{Br}_0(X)$ may well be infinite; the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ injects into, and is often equal to, $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$, which may be infinite if the divisible part of $\mathrm{Pic}(\overline{X})$ is non-zero, or if there is torsion in the Néron–Severi group $\mathrm{NS}(\overline{X})$; finally, $\mathrm{Br}(\overline{X})$ may be infinite. This prompts the following question.

Question 1. *Is* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *finite if* k *is finitely generated over its prime subfield?*

Let $\Gamma = \mathrm{Gal}(\overline{k}/k)$, and let $\mathrm{Br}(\overline{X})^\Gamma$ be the subgroup of Galois invariants of $\mathrm{Br}(\overline{X})$; then $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ naturally embeds into $\mathrm{Br}(\overline{X})^\Gamma$. A positive answer to Question 1 would follow from a positive answer to the following question.

Question 2. *Is* $\mathrm{Br}(\overline{X})^\Gamma$ *finite if* k *is finitely generated over its prime subfield?*

In this note we prove the following two theorems.

Theorem 1.1. *Let* k *be a field finitely generated over its prime subfield. Let* X *be a principal homogeneous space of an abelian variety over* k*.*

(i) *If the characteristic of* k *is* 0*, then* $\mathrm{Br}(\overline{X})^\Gamma$ *and* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *are finite.*

(ii) *If the characteristic of* k *is a prime* $p \neq 2$*, then* $\mathrm{Br}(\overline{X})^\Gamma(\mathrm{non}-p)$ *and* $(\mathrm{Br}(X)/\mathrm{Br}_1(X))(\mathrm{non}-p)$ *are finite.*

Theorem 1.2. *Let* k *be a field finitely generated over* $\mathbf{Q}$*. If* X *is a* $K3$ *surface over* k*, then the groups* $\mathrm{Br}(\overline{X})^\Gamma$ *and* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *are finite.*

Remark 1.3. The injective maps

$$\mathrm{Br}(X)/\mathrm{Br}_1(X) \hookrightarrow \mathrm{Br}(\overline{X})^\Gamma \quad \text{and} \quad \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \hookrightarrow \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$$

can be computed via the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(k, \mathrm{H}^q_{\acute{e}t}(\overline{X}, \mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\acute{e}t}(X, \mathbf{G}_m).$$

(A description of some of its differentials can be found in [26].) Recall that in characteristic zero the Picard group $\mathrm{Pic}\,(\overline{X})$ of a $K3$ surface X is a free abelian group of rank at most 20. The Galois group Γ acts on $\mathrm{Pic}\,(\overline{X})$ via a finite quotient, so that $\mathrm{H}^1(k, \mathrm{Pic}\,(\overline{X}))$ is finite. Thus in order to prove Theorem 1.2 it suffices to establish the finiteness of $\mathrm{Br}\,(\overline{X})^\Gamma$.

In the case when the rank of $\mathrm{Pic}\,(\overline{X})$ equals 20, Theorem 1.2 was proved by Raskind and Scharaschkin [23]. In an unpublished note, J.-L. Colliot-Thélène proved that $\mathrm{Br}\,(\overline{X})^\Gamma(\ell)$ is finite for every prime ℓ, where X is a smooth projective variety over a field finitely generated over $\mathbf{Q}$, assuming the Tate conjecture for divisors on X. (When $\dim(X) > 2$, he assumed additionally the semisimplicity of the Galois action on $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$.) See also [29] for some related results.

When X is an abelian variety over a field finitely generated over its prime subfield, the Tate conjecture for divisors on X (and the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$ for $\ell \neq p$) was proved by the second named author in characteristic $p > 2$ [37, 38], and by Faltings in characteristic zero [8, 9]. This result of Faltings combined with the construction of Kuga–Satake elaborated by Deligne [3], implies the Tate conjecture for divisors on $K3$ surfaces in characteristic zero [34, p. 80].

The novelty of our approach is due to the usage of a variant of the Tate conjecture for divisors on X [39, 41] which concerns the Galois invariants of the (twisted) second étale cohomology group with coefficients in $\mathbf{Z}/\ell$ (instead of $\mathbf{Q}_\ell$), for almost all primes ℓ. Using this variant we prove that under the conditions of Theorems 1.1 and 1.2 we have $\mathrm{Br}\,(\overline{X})^\Gamma_\ell = \{0\}$ for almost all primes ℓ.

Let k be a number field, $X(\mathbb{A}_k)$ the space of adelic points of X, and $X(\mathbb{A}_k)^{\mathrm{Br}}$ the subset of adelic points orthogonal to $\mathrm{Br}\,(X)$ with respect to the Brauer–Manin pairing (given by the sum of local invariants of an element of $\mathrm{Br}\,(X)$ evaluated at the local points; see [17]). We point out the following corollary to Theorem 1.2.

Corollary 1.4. *Let X be a $K3$ surface over a number field k. Then $X(\mathbb{A}_k)^{\mathrm{Br}}$ is an open subset of $X(\mathbb{A}_k)$.*

Proof. The sum of local invariants of a given element of $\mathrm{Br}\,(X)$ is a continuous function on $X(\mathbb{A}_k)$ with finitely many values. Thus the corollary is a consequence of Theorem 1.2. □

Let us mention here some open problems regarding rational points on $K3$ surfaces. Previous work on surfaces fibred into curves of genus 1 [2, 28, 27] indicates that it is not unreasonable to expect the Manin obstruction to be the only obstruction to the Hasse principle on $K3$ surfaces. One could raise a more daring question: is the set of k-points dense in the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$? By Corollary 1.4, this would imply that the set of k-points on any $K3$ surface over a number field is either empty or Zariski dense. Moreover, this would also imply the weak-weak approximation for $X(k)$, whenever this set is non-empty. (This means that k has a finite set of places S such that for any finite set of places T disjoint from S the diagonal image of $X(k)$ in $\prod_T X(k_v)$ is dense.)

The paper is organized as follows. In Section 2 we recall the basic facts about the interrelations between the Brauer group, the Picard group and the Néron–Severi group (mostly due to Grothendieck [11]). We also discuss some linear algebra constructions arising from ℓ-adic cohomology. In Section 3 we recall the finite coefficients variant of the Tate conjecture for abelian varieties and prove Theorem 1.1. Finally, Theorem 1.2 is proved in Section 4.

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2. The Néron–Severi group, H^2 and the Brauer group

We start with an easy lemma from linear algebra.

Lemma 2.1. *Let Λ be a principal ideal domain, H a non-zero Λ-module, $N \subset H$ a non-zero free submodule of finite rank. Let*

$$\psi : H \times H \to \Lambda$$

be a symmetric bilinear form. Let $N^\perp$ be the orthogonal complement to N in H with respect to ψ, and let δ be the discriminant of the restriction of ψ to N. If $\delta \neq 0$, then $N \cap N^\perp = \{0\}$ and

$$\delta^2 H \subset N \oplus N^\perp \subset H.$$

In particular, if δ is a unit in Λ, then $H = N \oplus N^\perp$.

Proof. Let us put $N^* = \mathrm{Hom}_\Lambda(N, \Lambda)$. The form ψ gives rise to a natural homomorphism of Λ-modules $e_\psi : H \to N^*$ with $N^\perp = \ker(e_\psi)$ and

$$\delta \cdot N^* \subset e_\psi(N) \subset N^*.$$

In particular, the restriction of e_ψ to N is injective; therefore $N \cap N^\perp = \{0\}$, and $e_\psi : N \to e_\psi(N)$ is an isomorphism. Let $u : e_\psi(N) \cong N$ be its inverse, i.e., $ue_\psi : N \to N$ is the identity map. Let us consider the homomorphism of Λ-modules

$$P : H \to N, \quad h \mapsto \delta u(e_\psi(h)).$$

This definition makes sense since $\delta e_\psi(h) \in \delta N^* \subset e_\psi(N)$. It is clear that $\delta \cdot \ker P \subset N^\perp \subset \ker(P)$, and P acts on N as the multiplication by δ. For any $h \in H$ we have $z = P(x) \in N$ and $P(z) = \delta z$, which implies that $P(\delta h) = P(z)$. Hence $\delta h - z \in \ker(P)$, and therefore $\delta(\delta h - z) \in N^\perp$. It follows that $\delta^2 h \in \delta z + N^\perp \subset N \oplus N^\perp$. □

2.2. Let us recall some useful elementary statements, which are due to Tate [31, 33]. Let B be an abelian group. The projective limit of the groups B_{ℓ^n} (where the transition maps are the multiplications by ℓ) is called the ℓ-adic Tate module of B and is denoted by $T_\ell(B)$. This limit carries a natural structure of a $\mathbf{Z}_\ell$-module; there is a natural injective map $T_\ell(B)/\ell \hookrightarrow B_\ell$. One may easily check that $T_\ell(B)_\ell = \{0\}$, and therefore $T_\ell(B)$ is torsion-free. Let us assume that B_ℓ is finite. Then all the B_{ℓ^n} are obviously finite, and $T_\ell(B)$ is finitely generated by Nakayama's lemma. Therefore, $T_\ell(B)$ is isomorphic to $\mathbf{Z}_\ell^r$ for some non-negative integer $r \le \dim_{\mathbf{F}_\ell}(B_\ell)$. Moreover, $T_\ell(B) = \{0\}$ if and only if $B(\ell)$ is finite.

For a field k with separable closure $\overline{k}$ we denote by Γ the Galois group $\mathrm{Gal}\,(\overline{k}/k)$. Let X be a geometrically integral smooth projective variety over k, and let $\overline{X} = X \times_k \overline{k}$.

Let $\ell \neq \mathrm{char}(k)$ be a prime. Following [11, II, Sect. 3] we recall that the exact Kummer sequence of sheaves in the étale topology,

$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \to \mathbf{G}_m \to 1,$$

gives rise to the (cohomological) exact sequence of Galois modules

$$0 \to \mathrm{Pic}\,(\overline{X})/\ell^n \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}\,(\overline{X})_{\ell^n} \to 0.$$

Since $\mathrm{Pic}\,(\overline{X})$ is an extension of the Néron–Severi group $\mathrm{NS}\,(\overline{X})$ by a divisible group, we have $\mathrm{Pic}\,(\overline{X})/\ell^n = \mathrm{NS}\,(\overline{X})/\ell^n$. We thus obtain the exact sequence of Galois modules

$$0 \to \mathrm{NS}\,(\overline{X})/\ell^n \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}\,(\overline{X})_{\ell^n} \to 0. \tag{1}$$

Since the groups $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})$ are finite, the groups $\mathrm{Br}\,(\overline{X})_{\ell^n}$ are finite as well [11, II, Cor. 3.4]. On passing to the projective limit we get an exact sequence of Γ-modules

$$0 \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \to T_\ell(\mathrm{Br}\,(\overline{X})) \to 0. \tag{2}$$

Since $T_\ell(\mathrm{Br}\,(\overline{X}))$ is a free $\mathbf{Z}_\ell$-module, this sequence shows that the torsion subgroup of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ is contained in $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$; that is, the torsion subgroups of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ and $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ coincide, and so are both equal to $\mathrm{NS}\,(\overline{X})(\ell)$. Tensoring the sequence (2) with $\mathbf{Q}_\ell$ (over $\mathbf{Z}_\ell$), we get the exact sequence of Γ-modules

$$0 \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1)) \to V_\ell(\mathrm{Br}\,(\overline{X})) \to 0, \tag{3}$$

where $V_\ell(\mathrm{Br}\,(\overline{X})) = T_\ell(\mathrm{Br}\,(\overline{X})) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. The Tate conjecture for divisors [30, 32, 34] asserts that if k is finitely generated over its prime subfield, then

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))^\Gamma = \mathrm{NS}\,(\overline{X})^\Gamma \otimes \mathbf{Q}_\ell. \tag{4}$$

Note also that (1) gives rise to the exact sequence of abelian groups

$$\begin{aligned} &0 \to (\mathrm{NS}\,(\overline{X})/\ell^n)^\Gamma \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})^\Gamma \to \mathrm{Br}\,(\overline{X})^\Gamma_{\ell^n} \\ &\to \mathrm{H}^1(k, \mathrm{NS}\,(\overline{X})/\ell^n) \to \mathrm{H}^1(k, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_{\ell^n})). \end{aligned} \tag{5}$$

The lemma that follows is probably well known; cf. [14, Sect. 5, pp. 16–17] and [7, pp. 198–199].

Lemma 2.3. *Let $L \in \mathrm{NS}\,(\overline{X})^\Gamma$ be a Galois invariant hyperplane section class. Assume that $d = \dim(X) \geq 2$. If $\mathrm{char}(k) = 0$, then the kernel of the symmetric intersection pairing*

$$\psi_0 : \mathrm{NS}\,(\overline{X}) \times \mathrm{NS}\,(\overline{X}) \to \mathbf{Z}, \quad x, y \mapsto x \cdot y \cdot L^{d-2},$$

is $\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$.

In any characteristic the same conclusion holds under the following condition:

> *there exist a finite extension k'/k with $k' \subset \overline{k}$, and a prime $q \neq \mathrm{char}(k)$ such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$, the $\mathrm{Gal}\,(\overline{k}/k')$-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ is semisimple, and $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))^{\mathrm{Gal}\,(\overline{k}/k')} = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$.*

Proof. We start with the case of characteristic zero. If K is an algebraically closed field containing k, then the Néron–Severi group $\mathrm{NS}\,(X \otimes_k K)$ is identified with the group of connected components of the Picard scheme of X [15, Cor. 4.18.3, Prop. 5.3, Prop. 5.10], and so does not depend on K. Let $k_0 \subset k$ be a subfield finitely generated over $\mathbf{Q}$, over which X and L are defined. Then there exists a smooth projective variety X_0 over the algebraic closure $\overline{k_0}$ of k_0 in $\overline{k}$, such that $\overline{X} = X_0 \times_{\overline{k_0}} \overline{k}$. The natural map $\mathrm{NS}\,(X_0) \to \mathrm{NS}\,(\overline{X})$ is bijective and therefore a group isomorphism.

For generalities on twisted classical cohomology groups we refer the reader to see [6, Sect. 1] or [4, Sect. 2.1].

Fix an embedding $\overline{k_0} \hookrightarrow \mathbf{C}$ and consider the complex variety $X_{\mathbf{C}} = X_0 \times_{\overline{k_0}} \mathbf{C}$. The natural map $\mathrm{NS}\,(X_0) \to \mathrm{NS}\,(X_{\mathbf{C}})$ is an isomorphism. Since the intersection indices do not depend on the choice of an algebraically closed ground field, it suffices to check the non-degeneracy of ψ_0 for the complex variety $X_{\mathbf{C}}$. In order to do so, consider the canonical embedding

$$\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q} \hookrightarrow \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)),$$

and the symmetric bilinear form

$$\rho: \ \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \times \ \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \to \ \mathbf{Q}, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The hard Lefschetz theorem says that the map

$$\mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1)) \longrightarrow \mathrm{H}^{2d-2}(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(d-1)), \quad x \mapsto x \cup L^{d-2},$$

is an isomorphism of vector spaces over $\mathbf{Q}$. Poincaré duality now implies that ρ is non-degenerate. Let us show that the restriction of ρ to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q} \subset \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ is also non-degenerate. Indeed, let $P \subset \mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ be the kernel of the multiplication by L^{d-1}. The group $\mathrm{H}^2(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}(1))$ is the orthogonal direct sum $\mathbf{Q}L \oplus P$. On the one hand, the form ρ is positive definite on $\mathbf{Q}L$ since L is ample. On the other hand, the restriction of ρ to P is negative definite, due to the Hodge–Riemann bilinear relations [35, Ch. V, Sect. 5, Thm. 5.3]. This implies the non-degeneracy of ρ on $\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}$, because this space is the direct sum of $\mathbf{Q}L$ and $(\mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}) \cap P$. To finish the proof, we note that the form induced by ρ on the Néron–Severi group coincides with ψ_0, whereas the kernel of $\mathrm{NS}\,(X_{\mathbf{C}}) \to \mathrm{NS}\,(X_{\mathbf{C}}) \otimes \mathbf{Q}$ is the torsion subgroup $\mathrm{NS}\,(X_{\mathbf{C}})_{\mathrm{tors}} = \mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$.

Now let us prove the lemma in the case of arbitrary characteristic, assuming the condition on the Galois module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$.

Let us replace k by k'. Consider the symmetric Galois-invariant $\mathbf{Q}_q$-bilinear form

$$\rho_q: \ \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \ \times \ \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \to \ \mathbf{Q}_q, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The hard Lefschetz theorem, proved by Deligne [5] in all characteristics, says that the map

$$h_L: \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \longrightarrow \mathrm{H}^{2d-2}_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(d-1)), \quad x \mapsto x \cup L^{d-2},$$

is an isomorphism of vector spaces over $\mathbf{Q}_q$. Thus h_L is an isomorphism of Galois modules. Poincaré duality now implies that ρ_q is non-degenerate.

Since h_L is an isomorphism of Galois modules, we have

$$\mathrm{H}^{2d-2}_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(d-1))^{\Gamma} = h_L(\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))^{\Gamma}) = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \cup L^{d-2}.$$

By the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ there is a unique Γ-invariant vector subspace W that is also a semisimple Γ-submodule such that

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \oplus W.$$

Our condition implies that $W^\Gamma = \{0\}$. If $M \subset \mathrm{H}^2_{ét}(\overline{X}, \mathbf{Q}_q(1))$ is a vector subspace that is also a simple Γ-submodule, and if $M \to \mathbf{Q}_q$ is a non-zero Γ-invariant linear form, then M is the trivial Γ-module $\mathbf{Q}_q$. It follows that the trivial Γ-module $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$ is orthogonal to W with respect to ρ_q. Now the non-degeneracy of ρ_q implies that its restriction

$$\psi_q : \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q \ \times \ \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q \to \mathbf{Q}_q, \quad x, y \mapsto x \cdot y \cdot L^{d-2},$$

is also non-degenerate. By the compatibility of the cohomology class of the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10], the bilinear form ψ_q is obtained from ψ_0 by tensoring it with $\mathbf{Q}_q$. To finish the proof we note that the kernel of $\mathrm{NS}\,(\overline{X}) \to \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$ is $\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}$. □

Remark 2.4. (i) Since $\mathrm{NS}\,(\overline{X})$ is a finitely generated abelian group, there exists a finite extension k'/k with $k' \subset \overline{k}$, such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$.

(ii) Recall that $V := \mathrm{H}^2_{ét}(\overline{X}, \mathbf{Q}_\ell(1))$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space. Let $G_{\ell,k}$ be the image of $\Gamma = \mathrm{Gal}\,(\overline{k}/k)$ in $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$; it is a compact subgroup of $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$ and, by the ℓ-adic version of Cartan's theorem [24], is an ℓ-adic Lie subgroup of $\mathrm{Aut}_{\mathbf{Q}_\ell}(V)$. If k'/k is a finite extension with $k' \subset \overline{k}$, then $\Gamma' = \mathrm{Gal}\,(\overline{k}/k')$ is an open subgroup of finite index in Γ; hence the image $G_{\ell,k'}$ of Γ' is an open subgroup of finite index in $G_{\ell,k}$. In particular, $G_{\ell,k}$ and $G_{\ell,k'}$ have the same Lie algebra, which is a $\mathbf{Q}_\ell$-Lie subalgebra of $\mathrm{End}_{\mathbf{Q}_\ell}(V)$. Applying Prop. 1 of [25], we conclude that V is semisimple as a $G_{\ell,k'}$-module if and only if it is semisimple as a $G_{\ell,k}$-module. It follows that $\mathrm{H}^2_{ét}(\overline{X}, \mathbf{Q}_\ell(1))$ is semisimple as a Γ'-module if and only if it is semisimple as a Γ-module.

The following statement was inspired by [11, III, Sect. 8, pp. 143–147] and [31, Sect. 5].

Proposition 2.5. *Let X be a smooth projective geometrically integral variety over a field k. Assume that one of the following conditions holds.*

(a) *X is a curve or a surface.*
(b) $\mathrm{char}(k) = 0$.
(c) *There exist a finite extension k'/k with $k' \subset \overline{k}$ and a prime $q \neq \mathrm{char}(k)$ such that $\mathrm{Gal}\,(\overline{k}/k')$ acts trivially on $\mathrm{NS}\,(\overline{X})$, the $\mathrm{Gal}\,(\overline{k}/k')$-module $\mathrm{H}^2_{ét}(\overline{X}, \mathbf{Q}_q(1))$ is semisimple, and $\mathrm{H}^2_{ét}(\overline{X}, \mathbf{Q}_q(1))^{\mathrm{Gal}\,(\overline{k}/k')} = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q$.*

Then for almost all primes ℓ the Γ-module $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ is a direct summand of the Γ-module $\mathrm{H}^2_{ét}(\overline{X}, \mathbf{Z}_\ell(1))$. If (c) *is satisfied, then $\mathrm{Br}\,(\overline{X})^\Gamma(q)$ is finite.*

Proof. (a) If X is a curve, then $\mathrm{H}^2_{ét}(\overline{X}, \mathbf{Z}_\ell(1)) = \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell \cong \mathbf{Z}_\ell$, and there is nothing to prove. Note that in this case $\mathrm{Br}\,(\overline{X}) = 0$ [11, III, Cor. 5.8]. Thus from now on we assume that $\dim(X) \geq 2$.

Let X be a surface, $n = |\mathrm{NS}\,(\overline{X})_{\mathrm{tors}}|$. The cycle map defines the commutative diagram of pairings given by the intersection index and the cup-product:

$$\begin{array}{ccccc} \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) & \times & \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) & \to & \mathbf{Z}_\ell \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{NS}\,(\overline{X}) & \times & \mathrm{NS}\,(\overline{X}) & \to & \mathbf{Z} \end{array} \tag{6}$$

The diagram commutes by the compatibility of the cohomology class of the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10]. The kernel of the pairing on the Néron–Severi group is its torsion subgroup. Let δ be the discriminant of the induced bilinear form on $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}}$. Let H be the Γ-module $\psi^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/_{\mathrm{tors}}$, and let ψ be the Galois-invariant $\mathbf{Z}_\ell$-bilinear form on H coming from the top pairing of (6). Let N be the Γ-submodule $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}} \otimes \mathbf{Z}_\ell \subset H$. It is clear that N is a free $\mathbf{Z}_\ell$-submodule of H, and δ is the discriminant of the restriction of ψ to N. Let $N^\perp$ be the orthogonal complement to N in H with respect to ψ; $N^\perp$ is obviously a Γ-submodule of H.

Applying Lemma 2.1 (with $\Lambda = \mathbf{Z}_\ell$) we conclude that

$$N \cap N^\perp = \{0\} \quad \text{and} \quad \delta^2 H \subset N \oplus N^\perp.$$

Now let $\tilde{M}$ be the preimage of $N^\perp$ in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. Clearly, $\tilde{M}$ is a Galois submodule, and $\tilde{M} \cap (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell)$ is the torsion subgroup of $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ and therefore coincides with $\mathrm{NS}\,(\overline{X})(\ell)$. It is also clear that

$$\delta^2\, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \subset (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) + \tilde{M}.$$

Let us put $M = n\tilde{M} \subset \tilde{M}$. We have

$$M \cap (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) = \{0\} \quad \text{and} \quad n\delta^2\, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \subset (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) \oplus M.$$

Since $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ is a finitely generated $\mathbf{Z}_\ell$-module, $(\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell) \oplus M$ is a subgroup of finite index in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. This index is 1 if ℓ does not divide $n\delta$.

(b) and (c). Let us choose a Γ-invariant hyperplane section class $L \in \mathrm{NS}\,(\overline{X})^\Gamma$. By Lemma 2.3 the symmetric bilinear form on $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}}$ induced by ψ_0 is non-degenerate. Let $\delta \in \mathbf{Z}$ be the discriminant of this form, $\delta \neq 0$. Let us consider the Galois-invariant symmetric $\mathbf{Z}_\ell$-bilinear form

$$\psi_1 : \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \times \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \to \mathbf{Z}_\ell, \quad x, y \mapsto x \cup y \cup L^{d-2}.$$

The compatibility of (the cohomology class of) the intersection of algebraic cycles and the cup-product of their cohomology classes [19, Ch. VI, Prop. 9.5 and Sect. 10] implies that the restriction of ψ_1 to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ coincides with the form induced by ψ_0. It follows from the hard Lefschetz theorem and Poincaré duality that $\ker(\psi_1) = \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_{\mathrm{tors}}$.

Let H be the Γ-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/_{\mathrm{tors}}$, and let ψ be the Galois-invariant $\mathbf{Z}_\ell$-bilinear form on H defined by ψ_1. Let N be the Γ-submodule $\mathrm{NS}\,(\overline{X})/_{\mathrm{tors}} \otimes \mathbf{Z}_\ell \subset H$. It is clear that N is a free $\mathbf{Z}_\ell$-submodule of H, and the discriminant of the restriction of ψ to N is δ. The rest of the proof is the same as in case (a).

Now suppose that under the condition of (c) the group $\mathrm{Br}\,(\overline{X})^\Gamma(q)$ is infinite. Since $\mathrm{Br}\,(\overline{X})^\Gamma \subset \mathrm{Br}\,(\overline{X})^{\mathrm{Gal}(\overline{k}/k')}$, we can extend the ground field from k to k'. For any n the group $\mathrm{Br}\,(\overline{X})_{q^n}$ is finite; thus there is an element of order q^n in $\mathrm{Br}\,(\overline{X})^\Gamma_{q^n}$ for every n; i.e., the set $S(n)$ of elements of order q^n in $\mathrm{Br}\,(\overline{X})^\Gamma_{q^n}$ is non-empty for all n. Since the projective limit of non-empty finite sets $S(n)$ is a non-empty subset of $T_q(\mathrm{Br}\,(\overline{X})^\Gamma) \setminus \{0\}$, we conclude that

$$T_q(\mathrm{Br}\,(\overline{X}))^\Gamma = T_q(\mathrm{Br}\,(\overline{X})^\Gamma) \neq \{0\}.$$

It follows that $V_q(\mathrm{Br}\,(\overline{X}))^\Gamma \neq \{0\}$. However, the semisimplicity of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1))$ implies that the exact sequence of Galois modules (3) splits; that is,

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_q(1)) \cong (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Q}_q) \oplus V_q(\mathrm{Br}\,(\overline{X})).$$

By condition (c) we have $V_q(\mathrm{Br}\,(\overline{X}))^\Gamma = \{0\}$. This contradiction proves the finiteness of $\mathrm{Br}\,(\overline{X})^\Gamma(q)$. □

Corollary 2.6. *Let X be a smooth projective geometrically integral variety over a field k. Assume that X/k satisfies one of the conditions* (a), (b), (c) *of Proposition* 2.5. *Then the map* $\mathrm{H}^1(k, \mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}/\ell) \to \mathrm{H}^1(k, \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell))$ *in* (5) *is injective for almost all ℓ.*

Proof. By Proposition 2.5, the Γ-module $\mathrm{NS}\,(\overline{X})/\ell = (\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell)/\ell$ is a direct summand of the Γ-module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell$ for almost all ℓ. We have an exact sequence

$$0 \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) \to \mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_\ell \to 0.$$

By a theorem of Gabber [10], for almost all ℓ the $\mathbf{Z}_\ell$-module $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell)$ has no torsion. Since $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell)$ and $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ are isomorphic as abelian groups, for almost all ℓ we have $\mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))_\ell = \{0\}$; hence $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) = \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))/\ell$. Thus $\mathrm{NS}\,(\overline{X})/\ell$ is a direct summand of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)$. This proves the corollary. □

Corollary 2.7. *Suppose that k is finitely generated over its prime subfield, and* $\mathrm{char}(k) \neq 2$. *Let A be an abelian variety over k. Then for all $\ell \neq$* $\mathrm{char}(k)$ *the subgroup* $\mathrm{Br}\,(\overline{A})^\Gamma(\ell)$ *is finite.*

Proof. Let ℓ be a prime different from $\mathrm{char}(k)$. The Tate conjecture for divisors (4) is true for any abelian variety A over such a field; in addition, the natural Galois action on the ℓ-adic cohomology groups of $\overline{A}$ is semisimple. (These assertions were proved by the second named author [37, 38] in finite characteristic not equal to 2, and by Faltings [8, 9] in characteristic zero.) This implies that A satisfies condition (c) of Proposition 2.5 for every prime $q \neq \mathrm{char}(k)$. Now the result follows from the last assertion of Proposition 2.5. $\square$

3. Proof of Theorem 1.1

Let A and A' be abelian varieties over an *arbitrary* field k. We write $\mathrm{Hom}(A, A')$ for the group of homomorphisms $A \to A'$. We have

$$\mathrm{Hom}(A, A') = \mathrm{Hom}_\Gamma(\overline{A}, \overline{A}') = \mathrm{Hom}(\overline{A}, \overline{A}')^\Gamma.$$

Since $\mathrm{Hom}(\overline{A}, \overline{A}')$ has no torsion, we have that $\mathrm{Hom}(A, A')/n$ is a subgroup of $\mathrm{Hom}(\overline{A}, \overline{A}')/n$.

Let A^t be the dual abelian variety of A. We have $(A^t)^t = A$ ([16, Ch. V, Sect. 2, Prop. 9], [21, p. 132]). Every divisor D on $\overline{A}$ defines the homomorphism $\overline{A} \to \overline{A}^t$ sending $a \in A(\overline{k})$ to the linear equivalence class of $T_a^*(D) - D$ in $\mathrm{Pic}^0(\overline{A})$, where T_a is the translation by a in A. If L is the algebraic equivalence class of D in $\mathrm{NS}\,(\overline{A})$, then this map depends only on L, and is denoted by $\phi_L : \overline{A} \to \overline{A}^t$ [21, Sect. 8]. For $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ we denote by $\alpha^t \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ the transpose of α. Note that $\phi_L^t = \phi_L$. Moreover, if we set

$$\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}} := \{u \in \mathrm{Hom}(\overline{A}, \overline{A}^t) \mid u = u^t\},$$

then the group homomorphism

$$\mathrm{NS}\,(\overline{A}) \to \mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}, \quad L \mapsto \phi_L,$$

is an isomorphism [16], [21, Sect. 20, formula (I) and Thm. 1 on p. 186, Thm. 2 on p. 188 and Remark on p. 189]. For any $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ we have $(\alpha^t)^t = \alpha$, and thus

$$\alpha + \alpha^t \in \mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}. \tag{7}$$

3.1. Let ℓ be a prime different from the characteristic of k, i a positive integer, and $n = \ell^i$. The kernel A_n of the multiplication by n in $A(\overline{k})$ is a Galois submodule, isomorphic to $(\mathbf{Z}/n)^{2\dim(A)}$ as an abelian group.

The natural map $\mathrm{Hom}(\overline{A}, \overline{A}')/n \to \mathrm{Hom}(A_n, A'_n)$ is *injective* [20, p. 124]. It commutes with the Galois action on both sides; in particular, the image of $\mathrm{Hom}(A, A')/n \subset \mathrm{Hom}(\overline{A}, \overline{A}')/n$ lies in $\mathrm{Hom}_\Gamma(A_n, A'_n)$.

For any $\alpha \in \mathrm{Hom}(\overline{A}, \overline{A}^t)$ and any $x, y \in A_n$ we have ([16, Ch. VII, Sect. 2, Thm. 4], [21, p. 186])

$$e_{n,A^t}(\alpha x, y) = e_{n,A}(x, \alpha^t y).$$

Thus $\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/n$ is a subgroup of

$$\mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}} := \{u \in \mathrm{Hom}(A_n, A_n^t) \mid e_{n,A^t}(ux, y) = e_{n,A}(x, uy), \quad \forall x, y \in A_n\}.$$

Moreover, if ℓ is odd, then, by (7), we have

$$\mathrm{Hom}(\overline{A}, \overline{A}^t)_{\mathrm{sym}}/n = \mathrm{Hom}(\overline{A}, \overline{A}^t)/n \cap \mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}. \tag{8}$$

Remark 3.2. The two (non-degenerate, Galois-equivariant) Weil pairings

$$e_{n,A} : A_n \times A_n^t \to \mu_n \quad \text{and} \quad e_{n,A^t} : A_n^t \times A_n \to \mu_n$$

differ by -1 [16, Ch. VII, Sect. 2, Thm. 5(iii) on p. 193]; that is,

$$e_{n,A^t}(y, x) = -e_{n,A}(x, y)$$

for all $x \in A_n$, $y \in A_n^t$. Since for each $u \in \mathrm{Hom}(A_n, A_n^t)$ we have

$$e_{n,A}(x, uy) = -e_{n,A^t}(uy, x) = -e_{n,A}(y, u^t x),$$

we conclude that u lies in $\mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}$ if and only if the bilinear form $e_{n,A}(x, uy)$ is *skew-symmetric*; that is, for any $x, y \in A_n$ we have

$$e_{n,A}(x, uy) = -e_{n,A}(y, ux).$$

3.3. For a module M over a commutative ring Λ we denote by $\mathrm{S}^2_\Lambda M$ the submodule of $M \otimes_\Lambda M$ generated by $x \otimes x$ for all $x \in M$. Let $\wedge^2_\Lambda M = (M \otimes_\Lambda M)/\mathrm{S}^2_\Lambda M$. We have $x \otimes y + y \otimes x \in \mathrm{S}^2_\Lambda M$; these elements generate $\mathrm{S}^2_\Lambda M$ if 2 is invertible in Λ.

From the Kummer sequence one obtains the well-known canonical isomorphism $\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mu_n) = \mathrm{Pic}\,(\overline{A})_n = A_n^t$. Thus we have canonical isomorphisms of Galois modules (cf. [1, Sect. 2], [19], [20]):

$$\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_n) = \wedge^2_{\mathbf{Z}/n} A_n^t(-1) = \mathrm{Hom}(\wedge^2_{\mathbf{Z}/n} A_n, \mu_n).$$

Clearly, there is a canonical embedding of Galois modules

$$\mathrm{Hom}(\wedge^2_{\mathbf{Z}/n} A_n, \mu_n) \hookrightarrow \mathrm{Hom}(A_n, A_n^t),$$

whose image coincides with the set of $u : A_n \to A_n^t$ such that the bilinear form $e_{n,A}(x, uy)$ is *alternating*, i.e., $e_{n,A}(x, ux) = 0$ for all $x \in A_n$. Combining it with Remark 3.2, we conclude that if ℓ is *odd*, then there is a canonical isomorphism of Galois modules

$$\mathrm{H}^2_{\acute{e}t}(\overline{A}, \mu_n) \cong \mathrm{Hom}(A_n, A_n^t)_{\mathrm{sym}}. \tag{9}$$

Let us recall a variant of the Tate conjecture on homomorphisms that first appeared in [39].

Proposition 3.4. *Let k be a field finitely generated over its prime subfield,* $\mathrm{char}(k)\neq 2$*. If A and A' are abelian varieties over k, then the natural injection*

$$\mathrm{Hom}(A,A')/\ell \hookrightarrow \mathrm{Hom}_\Gamma(A_\ell,A'_\ell) \tag{10}$$

is an isomorphism for almost all ℓ.

Proof. In the finite characteristic case this is proved in [39, Thm. 1.1]. When $A=A'$ and k is a number field, Cor. 5.4.5 of [41] (based on the results of Faltings [8]) says that for almost all ℓ we have

$$\mathrm{End}(A)/\ell = \mathrm{End}_\Gamma(A_\ell). \tag{11}$$

The same proof works over arbitrary fields that are finitely generated over $\mathbf{Q}$, provided one replaces the reference to Prop. 3.1 of [41] by the reference to the corollary on p. 211 of Faltings [9]. Applying (11) to the abelian variety $A\times A'$, we deduce that (10) is a bijection. □

Lemma 3.5. *Let k be a field finitely generated over its prime subfield,* $\mathrm{char}(k)\neq 2$*, and let A be an abelian variety over k. Then for almost all ℓ we have the following statements:*

(i) *the injective map* $(\mathrm{NS}\,(\overline{A})/\ell)^\Gamma \hookrightarrow \mathrm{H}^2_{\acute{e}t}(\overline{A},\mu_\ell)^\Gamma$ *in* (5) *is an isomorphism;*

(ii) $\mathrm{Br}\,(\overline{A})^\Gamma(\ell)=\{0\}$.

Proof. Suppose that ℓ is odd. By (8) we have

$$\mathrm{Hom}(\overline{A},\overline{A}^t)_{\mathrm{sym}}/\ell = \mathrm{Hom}(\overline{A},\overline{A}^t)/\ell \cap \mathrm{Hom}(A_\ell,A^t_\ell)_{\mathrm{sym}}.$$

Proposition 3.4 implies that for almost all ℓ we have

$$\mathrm{Hom}(A,A^t)/\ell = \mathrm{Hom}(A_\ell,A^t_\ell)^\Gamma = \mathrm{Hom}_\Gamma(A_\ell,A^t_\ell).$$

We thus obtain an isomorphism

$$\mathrm{Hom}(A,A^t)_{\mathrm{sym}}/\ell = \mathrm{Hom}_\Gamma(A_\ell,A^t_\ell)_{\mathrm{sym}}. \tag{12}$$

The left hand side of (12) is $\mathrm{Hom}_\Gamma(\overline{A},\overline{A}^t)_{\mathrm{sym}}/\ell \cong \mathrm{NS}\,(\overline{A})^\Gamma/\ell$; see the beginning of this section. The right hand side of (12) is isomorphic to $\mathrm{H}^2_{\acute{e}t}(\overline{A},\mu_\ell)^\Gamma$ by (9). It follows that $\mathrm{NS}\,(\overline{A})^\Gamma/\ell$ and $\mathrm{H}^2_{\acute{e}t}(\overline{A},\mu_\ell)^\Gamma$ have the same number of elements. Since $\mathrm{NS}\,(\overline{A})$ has no torsion, $\mathrm{NS}\,(\overline{A})^\Gamma/\ell$ is a subgroup of $(\mathrm{NS}\,(\overline{A})/\ell)^\Gamma$, and hence the injective map in (i) is bijective. Statement (ii) follows from (i), Corollary 2.6 and the exact sequence (5). □

End of proof of Theorem 1.1. Let A be an abelian variety over k, and X a principal homogeneous space of A. In characteristic 0 (resp. in characteristic p) it suffices to show that $\mathrm{Br}\,(\overline{X})^\Gamma$ (resp. $\mathrm{Br}\,(\overline{X})^\Gamma(\text{non}-p)$) is finite. For this we can go over to a finite extension k'/k such that $X \times_k k' \simeq A \times_k k'$, and so assume that $X = A$. The theorem now follows from Lemma 3.5 (ii) and Corollary 2.7. □

4. Proof of Theorem 1.2

4.1. In this subsection we recall some well-known results which will be used later in this section.

Let A be an abelian variety over a field k, ℓ a prime different from $\mathrm{char}(k)$, $n = \ell^i$. Let $\pi_1^{ét}(\overline{A}, 0)^{(\ell)}$ be the maximal abelian ℓ-quotient of the Grothendieck étale fundamental group $\pi_1^{ét}(\overline{A}, 0)$. Let us consider the Tate ℓ-module $T_\ell(A) := T_\ell(A(\overline{k}))$. It is well known [16, 21] that $T_\ell(A)$ is a free $\mathbf{Z}_\ell$-module of rank $2\dim(A)$ equipped with a natural structure of a Γ-module, and the natural map $T_\ell(A)/n \to A_n$ is an isomorphism of Galois modules. Recall [20, pp. 129–130] that the isogeny $\overline{A} \xrightarrow{n} \overline{A}$ is a Galois étale covering with the Galois group A_n acting by translations. This defines a canonical surjection $f_n : \pi_1^{ét}(\overline{A}, 0)^{(\ell)} \twoheadrightarrow A_n$. The f_n glue together into a canonical isomorphism of Galois modules $\pi_1^{ét}(\overline{A}, 0)^{(\ell)} \to T_\ell(A)$, which induces the canonical isomorphisms of Galois modules

$$\mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\pi_1^{ét}(\overline{A}, 0)^{(\ell)}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell).$$

Since $\mathrm{H}^j_{ét}(\overline{A}, \mathbf{Z}_\ell)$ is torsion-free for any j [20, Thm. 15.1(b) on p. 129], the reduction modulo n gives rise to natural isomorphisms of Galois modules

$$\mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}/n) = \mathrm{H}^1_{ét}(\overline{A}, \mathbf{Z}_\ell)/n = \mathrm{Hom}(A_n, \mathbf{Z}/n).$$

Now suppose that we are given a field embedding $\overline{k} \hookrightarrow \mathbf{C}$. Let us consider the complex abelian variety $B = A(\mathbf{C})$. The exponential map establishes a canonical isomorphism of compact Lie groups $\mathrm{Lie}(B)/\Pi \to B$ [21, Sect. 1]. Here $\mathrm{Lie}(B) \cong \mathbf{C}^{\dim(B)}$ is the tangent space to B at the origin, Π is a discrete lattice of rank $2\dim(B)$, and the natural map $\mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{R} \to \mathrm{Lie}(B)$ is an isomorphism of real vector spaces. Clearly, V is the universal covering space of B, and the fundamental group $\pi_1(B, 0) = \mathrm{H}_1(B, \mathbf{Z}) = \Pi$ acts on V by translations. We have

$$B_n = \frac{1}{n}\Pi/\Pi \subset V/\Pi = B.$$

The isogeny $B \xrightarrow{n} B$ is an unramified Galois covering of connected spaces (in the classical topology) with the Galois group B_n, corresponding to the subgroup $n\Pi \subset \Pi$. It is identified with $V/n\Pi \to V/\Pi$, and the corresponding homomorphism $\varphi_n : \Pi \twoheadrightarrow B_n = \frac{1}{n}\Pi/\Pi$ sends c to $\frac{1}{n}c + \Pi$. The comparison theorem for fundamental groups implies that φ_n coincides with the composition

$$\pi_1(B,0) \to \pi_1^{ét}(B,0) \to \pi_1^{ét}(B,0)^{(\ell)} \xrightarrow{f_n} B_n.$$

We obtain the following sequence of homomorphisms:

$$\begin{aligned} &\mathrm{Hom}(B_n, \mathbf{Z}/n) \hookrightarrow \mathrm{Hom}(\pi_1^{ét}(B,0)^{(\ell)}, \mathbf{Z}/n) \\ &= \mathrm{Hom}(\pi_1^{ét}(B,0), \mathbf{Z}/n) \to \mathrm{Hom}(\pi_1(B,0), \mathbf{Z}/n). \end{aligned} \tag{13}$$

The same comparison theorem implies that the last map in (13) is bijective. It follows easily that all the homomorphisms in (13) are isomorphisms. Recall that

$$\mathrm{Hom}(\pi_1^{ét}(B,0)^{(\ell)}, \mathbf{Z}/n) = \mathrm{H}^1_{ét}(B, \mathbf{Z}/n), \quad \mathrm{Hom}(\pi_1(B,0), \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z}/n).$$

Note also that φ_n establishes a canonical isomorphism

$$\Pi/n = \pi_1(B,0)/n \to B_n, \quad c \mapsto \frac{1}{n}c + \Pi,$$

which gives us the canonical isomorphisms

$$\mathrm{H}^1(B, \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z})/n = \mathrm{Hom}(B_n, \mathbf{Z}/n) = \mathrm{H}^1_{ét}(B, \mathbf{Z}/n).$$

Taking the projective limits with respect to i (recall that $n = \ell^i$), we get the canonical isomorphisms

$$\mathrm{H}^1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(B), \mathbf{Z}_\ell) = \mathrm{H}^1_{ét}(B, \mathbf{Z}_\ell).$$

On the other hand, taking the projective limit of the φ_n, we get the natural map [21, p. 237]

$$\mathrm{H}_1(B, \mathbf{Z}) = \Pi \to T_\ell(B), \quad x \mapsto \{x/\ell^i\}_{i=1}^{\infty},$$

which extends by $\mathbf{Z}_\ell$-linearity to the natural isomorphism of $\mathbf{Z}_\ell$-modules

$$\varphi^{(\ell)} : \mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell = \Pi \otimes \mathbf{Z}_\ell \cong T_\ell(B).$$

We have

$$A_n = B_n = \mathrm{H}_1(B, \mathbf{Z})/n. \tag{14}$$

The comparison theorem for étale and classical cohomology implies that $\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}/n) = \mathrm{H}^1(B, \mathbf{Z}/n)$; thus we obtain

$$
\begin{aligned}
(15) \qquad \mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}/n) = \mathrm{Hom}(A_n, \mathbf{Z}/n) = \\
\mathrm{Hom}(B_n, \mathbf{Z}/n) = \mathrm{Hom}(\mathrm{H}_1(B, \mathbf{Z})/n, \mathbf{Z}/n), \\
T_\ell(A) = T_\ell(B) = \mathrm{H}_1(B, \mathbf{Z}) \otimes \mathbf{Z}_\ell, \\
\mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell) = \mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{H}^1_{\acute{e}t}(B, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(B), \mathbf{Z}_\ell).
\end{aligned}
$$

Lemma 4.2. *Let M and N be subgroups of $\mathbf{Z}^n$ such that $M \cap N = 0$. Then for almost all ℓ the natural maps $M/\ell \to (\mathbf{Z}/\ell)^n$ and $N/\ell \to (\mathbf{Z}/\ell)^n$ are injective, and the intersection of their images is $\{0\}$.*

Proof. There is a subgroup $L \subset \mathbf{Z}^n$ such that $L \cap (M \oplus N) = 0$, and $L \oplus M \oplus N$ is of finite index in $\mathbf{Z}^n$. For all ℓ not dividing this index, the canonical map $M/\ell \to (\mathbf{Z}/\ell)^n$ and the similar map for N are injective. Moreover, $(\mathbf{Z}/\ell)^n$ is the direct sum of L/ℓ, M/ℓ and N/ℓ. This proves the lemma. □

Lemma 4.3. *Let X be a $K3$ surface over a field k finitely generated over $\mathbf{Q}$. Then the injective map $(\mathrm{NS}\,(\overline{X})/\ell)^\Gamma \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)^\Gamma$ in (5) is an isomorphism for almost all primes ℓ.*

Proof. It suffices to prove the lemma for a finite extension k'/k, $k' \subset \overline{k}$, and $\Gamma' = \mathrm{Gal}\,(\overline{k}/k') \subset \Gamma$. Indeed, for any Γ-module M the composition of the natural inclusion $M^\Gamma \hookrightarrow M^{\Gamma'}$ and the norm map $M^{\Gamma'} \to M^\Gamma$ is the multiplication by the degree $[k' : k]$. Hence if $(\mathrm{NS}\,(\overline{X})/\ell)^{\Gamma'} \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)^{\Gamma'}$ is surjective for all primes ℓ not dividing a certain integer N, then so is the original map $(\mathrm{NS}\,(\overline{X})/\ell)^\Gamma \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)^\Gamma$ for all primes ℓ not dividing $N[k' : k]$. In particular, we can assume without loss of generality that Γ acts trivially on $\mathrm{NS}\,(\overline{X})$.

Now let us fix an embedding $\overline{k} \hookrightarrow \mathbf{C}$ and identify $\overline{k}$ with its image in $\mathbf{C}$.

The group $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)) \simeq \mathbf{Z}^{22}$ has a natural $\mathbf{Z}$-valued bilinear form ψ given by the intersection index. By Poincaré duality ψ is *unimodular*, i.e., the map $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)) \to \mathrm{Hom}(\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)), \mathbf{Z})$ induced by ψ is an isomorphism. Since $X(\mathbf{C})$ is simply connected we have $\mathrm{H}^1(X(\mathbf{C}), \mathbf{Z}) = \{0\}$, and by Poincaré duality this implies $\mathrm{H}^3(X(\mathbf{C}), \mathbf{Z}) = \{0\}$. Recall that $\mathrm{NS}\,(\overline{X}) = \mathrm{NS}\,(X_\mathbf{C})$ (see the beginning of the proof of Lemma 2.3). Since $X(\mathbf{C})$ is simply connected we have

$$\mathrm{Pic}\,(X_\mathbf{C}) = \mathrm{NS}\,(X_\mathbf{C}) = \mathrm{NS}\,(\overline{X}) = \mathrm{Pic}\,(\overline{X}).$$

We define the lattice of transcendental cycles T_X as the orthogonal complement to the injective image of $\mathrm{NS}\,(\overline{X})$ in $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))$. The restriction of ψ to $\mathrm{NS}\,(\overline{X})$ is non-degenerate, and we write δ for the absolute value of the corresponding discriminant. Then $\mathrm{NS}\,(\overline{X}) \cap T_X = 0$, and $\mathrm{NS}\,(\overline{X}) \oplus T_X$ is a subgroup of $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))$ of finite index δ. Let ℓ be a prime not dividing δ. Then we have

$$\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))/\ell = (\mathrm{NS}\,(\overline{X})/\ell) \oplus (T_X/\ell).$$

The restriction of the $\mathbf{Z}/\ell$-valued pairing induced by ψ to $\mathrm{NS}\,(\overline{X})/\ell$ is a non-degenerate $\mathbf{Z}/\ell$-bilinear form, so that T_X/ℓ is the orthogonal complement to $\mathrm{NS}\,(\overline{X})/\ell$. Since $\mathrm{H}^3(X(\mathbf{C}), \mathbf{Z}) = \{0\}$, we have $\mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))/\ell = \mathrm{H}^2(X(\mathbf{C}), \mu_\ell)$. The comparison theorem gives an isomorphism of $\mathbf{Z}_\ell$-modules

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) = \mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1)) \otimes \mathbf{Z}_\ell, \tag{16}$$

which is compatible with cup-products [7, Prop. 6.1, p. 197], [6, Example 2.1(b), pp. 28–29]. Reducing modulo ℓ we get an isomorphism of $\mathbf{Z}/\ell$-vector spaces $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) = \mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))/\ell$, compatible with cup-products. Thus for ℓ not dividing δ we have an orthogonal direct sum

$$\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell) = (\mathrm{NS}\,(\overline{X})/\ell) \oplus (T_X/\ell),$$

so that for these ℓ the abelian group T_X/ℓ carries a natural Γ-(sub)module structure. (Here we use the compatibility of the cycle maps $\mathrm{Pic}\,(\overline{X}) \to \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)$ and $\mathrm{Pic}\,(\overline{X}) \to \mathrm{H}^2(X(\mathbf{C}), \mu_\ell)$; see [13, Prop. 3.8.5, pp. 296–297].)

Let $L \in \mathrm{Pic}\,(\overline{X}) = \mathrm{NS}\,(\overline{X})$ be a Γ-invariant hyperplane section class, and $P \subset \mathrm{H}^2(X(\mathbf{C}), \mathbf{Z}(1))$ the orthogonal complement to L with respect to ψ. Then (16) implies that $P_\ell = P \otimes \mathbf{Z}_\ell$ is both a Galois and a $\mathbf{Z}_\ell$-submodule of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$. It is clear that P_ℓ is the orthogonal complement to L in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ with respect to the Galois-invariant intersection pairing

$$\psi_\ell : \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \times \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1)) \to \mathbf{Z}_\ell.$$

Similarly, $T_X \otimes \mathbf{Z}_\ell$ is the orthogonal complement to $\mathrm{NS}\,(\overline{X}) \otimes \mathbf{Z}_\ell$ in $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Z}_\ell(1))$ with respect to ψ_ℓ, and so is a Galois submodule.

Let $C^+(P)$ be the even Clifford $\mathbf{Z}$-algebra of (P, ψ). The complex vector space $P_\mathbf{C} := P \otimes \mathbf{C}$ inherits from $\mathrm{H}^2(X(\mathbf{C}), \mathbf{C}(1))$ the Hodge decomposition of type $\{(-1,1),(0,0),(1,-1)\}$ with Hodge numbers $h^{1,-1} = h^{-1,1} = 1$. By the Lefschetz theorem, T_X intersects trivially with the $(0,0)$-subspace. The $\mathbf{Z}$-algebra $C^+(P)$ naturally carries a weight zero Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$ induced by the Hodge structure on P (via the identification $C^+(P) = \bigoplus_i \wedge^{2i} P$); see [3, Lemma 4.4]. On the other hand, $C^+(P) \otimes \mathbf{Z}_\ell$ coincides with the even Clifford $\mathbf{Z}_\ell$-algebra $C^+(P_\ell)$ of (P_ℓ, ψ_ℓ). Clearly, $C^+(P_\ell)$ carries a natural Γ-module structure induced by the Galois action on P_ℓ (via the identification $C^+(P_\ell) = \bigoplus_i \wedge^{2i}_{\mathbf{Z}_\ell} P_\ell$). In his adaptation of the Kuga–Satake construction, Deligne ([3], pp. 219–223, in particular Prop. 5.7 and Lemma 6.5.1; see also [22] and [7], pp. 218–219) shows that after replacing k by a finite extension, there exists an abelian variety A over k and an injective ring homomorphism

$$u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}))$$

satisfying the following properties.

(a) $u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}))$ is a morphism of weight zero Hodge structures.

(b) The $\mathbf{Z}_\ell$-algebra homomorphism

$$u_\ell : C^+(P_\ell) \hookrightarrow \mathrm{End}_{\mathbf{Z}}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z})) \otimes \mathbf{Z}_\ell = \mathrm{End}_{\mathbf{Z}_\ell}(\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell))$$

obtained from u by tensoring it with $\mathbf{Z}_\ell$, and then applying the comparison isomorphism $\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_\ell = \mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)$, is an injective homomorphism of Galois modules.

Replacing, if necessary, k by a finite extension we may and will assume that all the endomorphisms of $\overline{A}$ are defined over k, that is, $\mathrm{End}(A) = \mathrm{End}(\overline{A})$.

Using the compatible isomorphisms (see Subsection 4.1)

$$\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z}) = \mathrm{Hom}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}), \mathbf{Z}),$$

$$\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), \mathbf{Z}_\ell), \quad T_\ell(A) = \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}) \otimes \mathbf{Z}_\ell,$$

we obtain the compatible ring anti-isomorphisms

$$t : \mathrm{End}(\mathrm{H}^1(A(\mathbf{C}), \mathbf{Z})) \cong \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})),$$

$$t_\ell : \mathrm{End}_{\mathbf{Z}_\ell}(\mathrm{H}^1_{\acute{e}t}(\overline{A}, \mathbf{Z}_\ell)) \cong \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A))$$

of weight zero Hodge structures and Galois modules, respectively. Taking the compositions, we get an injective homomorphism of weight zero Hodge structures

$$t\, u : C^+(P) \hookrightarrow \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})),$$

which, extended by $\mathbf{Z}_\ell$-linearity, coincides with the injective homomorphism of Galois modules

$$t_\ell\, u_\ell : C^+(P_\ell) \hookrightarrow \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A)).$$

We shall identify $C^+(P)$ and $\mathrm{End}(A)$ with their images in $\mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}))$. Note that all the elements of $\mathrm{End}(A) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z}))$ have pure Hodge type $(0, 0)$.

Let us first consider the case when $\mathrm{rk}\,\mathrm{NS}\,(\overline{X}) \geq 2$. Then there exists a non-zero element $m \in \mathrm{NS}\,(\overline{X})^\Gamma \cap P$. Then

$$m \wedge T_X \subset \wedge^2 P \subset C^+(P) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})).$$

Since $m \wedge T_X$ does not contain non-zero elements of type $(0,0)$, we have

$$(m \wedge T_X) \cap \mathrm{End}(A) = 0.$$

Using (14) and (15), we observe that for all but finitely many ℓ the Γ-module T_X/ℓ is isomorphic to

$$(m \wedge T_X)/\ell \subset \mathrm{End}_{\mathbf{Z}_\ell}(T_\ell(A))/\ell = \mathrm{End}_{\mathbf{F}_\ell}(A_\ell).$$

Lemma 4.2 implies that $(m \wedge T_X)/\ell$ intersects trivially with $\mathrm{End}(A)/\ell$ for almost all ℓ. By the variant of the Tate conjecture (Proposition 3.4), for almost all ℓ we have $\mathrm{End}_{\mathbf{F}_\ell}(A_\ell)^\Gamma = \mathrm{End}_\Gamma(A_\ell) = \mathrm{End}(A)/\ell$; thus every Γ-invariant element of $m \wedge (T_X/\ell)$ is contained in $\mathrm{End}(A)/\ell$, and hence must be zero. It follows that $(T_X/\ell)^\Gamma = 0$ for almost all ℓ. Therefore, $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)^\Gamma = (\mathrm{NS}\,(\overline{X})/\ell)^\Gamma$ for almost all ℓ.

It remains to consider the case $\mathrm{rk}\,\mathrm{NS}\,(\overline{X}) = 1$. Then $T_X = P \simeq \mathbf{Z}^{21}$, and so $\wedge^{20} T_X$ is the dual lattice of T_X. We have

$$\wedge^{20} T_X = \wedge^{20} P \subset C^+(P) \subset \mathrm{End}(\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})).$$

Since T_X does not contain non-zero elements of type $(0,0)$, the same is true for the dual Hodge structure $\wedge^{20} T_X$. Thus $\wedge^{20} T_X \cap \mathrm{End}(A) = 0$, and the same arguments as before show that $(\wedge^{20} T_X/\ell)^\Gamma = 0$ for almost all ℓ. The bilinear $\mathbf{Z}/\ell$-valued form induced by the cup-product on $T_X/\ell \subset \mathrm{H}^2_{\acute{e}t}(\overline{X}, \mu_\ell)$ is non-degenerate for almost all ℓ, so that this Galois module is self-dual. Thus the Galois modules T_X/ℓ and $\wedge^{20} T_X/\ell$ are isomorphic, and we conclude that $(T_X/\ell)^\Gamma = 0$. This finishes the proof. □

Lemma 4.4. *Let X be a $K3$ surface over a field k finitely generated over $\mathbf{Q}$. Then $\mathrm{Br}\,(\overline{X})^\Gamma(\ell)$ is finite for all ℓ.*

Proof. By Proposition 2.5, it suffices to check the validity of the Tate conjecture for divisors and the semisimplicity of the Galois module $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell(1))$. Both these assertions follow from the corresponding results on abelian varieties, proved by Faltings in [8, 9]. The latter follows from the semisimplicity of the Galois action on the ℓ-adic cohomology groups of abelian varieties combined with Proposition 6.26(d) of [7]. The former follows from the validity of the Tate conjecture for divisors on abelian varieties, as explained on p. 80 of [34]. □

End of proof of Theorem 1.2. By Remark 1.3, it suffices to show that $\mathrm{Br}\,(\overline{X})^\Gamma$ is finite. By the exact sequence (5), Corollary 2.6 and Lemma 4.3 we have $\mathrm{Br}\,(\overline{X})_\ell^\Gamma = 0$ for almost all ℓ. Now the finiteness of $\mathrm{Br}\,(\overline{X})^\Gamma$ follows from Lemma 4.4. □

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