M2P4 Rings and Fields
Test 1, solutions.

30 January 2008

1. **6 marks**

The zero divisors in $\mathbb{Z}/2008$ are the residue classes of all the integers that have a common factor with 2008 (except 2008 itself). (3 marks for this.) These are the even numbers from 2 to 2006, and also the residue classes of the multiples of 251, i.e. $\overline{251n}$, where $n = 1, 2, 3, 4, 5, 6, 7$.

2. **6 marks**

Write $29 = (a + b\sqrt{-7})(c + d\sqrt{-7})$. Then we look for solutions of $29 = a^2 + 7b^2$ with $b \neq 0$, and so find that $29 = (1 + 2\sqrt{-7})(1 - 2\sqrt{-7})$. Neither factor is a unit, since $\mathbb{Z}[\sqrt{-7}]^* = \{\pm 1\}$.

3. **8 marks**

We have $e^{\frac{\pi i}{4}} = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2}(1 + i)$, so that the minimal polynomial over $\mathbb{R}$ is $x^2 - \sqrt{2}x + 1 = 0$. (3 marks for this.)

We have $(e^{\frac{\pi i}{4}})^4 = -1$, so our number is a root of $x^4 + 1 = 0$. The roots of $x^4 + 1$ are $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$ and their conjugates. The polynomial $x^4 + 1$ is divisible by $x^2 - \sqrt{2}x + 1$ by the first part of the question. A monic polynomial with real coefficients that vanishes at $e^{\frac{\pi i}{4}}$ and divides $x^4 + 1$ is either $x^2 - \sqrt{2}x + 1$ or $x^4 + 1$. The first of these has an irrational coefficient. Thus the answer is $x^4 + 1$. (5 marks. 2 marks for the correct answer without proof.)