M2P4 Rings and Fields
Test 2, solutions.

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1.

(a) **4 marks**

It's enough to check that if x, $y \in I \cap J$, then $x + y$, $-x \in I \cap J$, and $rx \in I \cap J$ for any $r \in R$. Indeed, x, $y \in I$ hence $x + y$, $-x$, $rx \in I$, and similarly for J. Thus $x + y$, $-x$, $rx \in I \cap J$.

(b) **6 marks**

Recall that $\mathbb{Q}[x]$ is a UFD, so that every element is uniquely written as a product of irreducible factors. $I \cap J$ consists of all the polynomials divisible by both $x^2 - 1 = (x - 1)(x + 1)$ and $x^3 + 1 = (x + 1)(x^2 - x + 1)$, where the factors are irreducible ($x^2 - x + 1$ has no roots in $\mathbb{Q}$, and so is irreducible). Thus $I \cap J$ consists of all polynomials divisible by the least common multiple of $x^2 - 1$ and $x^3 + 1$, which is $(x - 1)(x^3 + 1) = x^4 - x^3 + x - 1$. Therefore, $I \cap J = (x^4 - x^3 + x - 1)\mathbb{Q}[x]$.

2. **10=2+2+2+4 marks**

no, yes, no, no

The elements of $\mathbb{Z}/5$ are $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}$. Their squares are $\overline{0}, \overline{1}, \overline{4}, \overline{4}, \overline{1}$, respectively. Their cubes are $\overline{0}, \overline{1}, \overline{3}, \overline{2}, \overline{4}$, respectively. Hence $x^2 + \overline{1} = (x + \overline{2})(x + \overline{3})$ is reducible, whereas $x^2 + x + 1$ is not since it has no roots (or since its discriminant $-\overline{3}$ is not a square in $\mathbb{Z}/5$). Note that $\overline{2}$ is a root of the polynomial $x^3 + \overline{2}$, which is thus reducible. Finally, $x^2 + \overline{1} = (x + \overline{2})(x + \overline{3})$ implies $x^4 + \overline{1} = (x^2 + \overline{2})(x^2 + \overline{3})$, so the last polynomial is reducible.