Rings and Fields
Test 2.

20 February 2008

1. **8 marks**, if all details are given.

We proved that any ED is a PID, and any PID is a UFD. 2 is an irreducible element of $\mathbb{Z}[\sqrt{-5}]$, because $2 = (a + b\sqrt{-5})(c + d\sqrt{-5})$, where the factors are non-units, implies $2 = a^2 + 5b^2$ which has no solutions in integers. Now 2 divides $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ but does not divide either factor. Hence $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, hence not a ED.

2. **12 marks**, 2 marks for each part.

$\mathbb{R}$ is a field, so the zero ideal is maximal.

$\mathbb{R}[t]$ is a PID, hence the maximal ideals are generated by irreducible polynomials. So the ideals are $(x + a)$, $a \in \mathbb{R}$, and $(x^2 + bx + c)$, $b, c \in \mathbb{R}$, $b^2 - 4c < 0$.

$\mathbb{C}[t]$ is a PID, hence the maximal ideals are generated by irreducible polynomials. So the ideals are $(x + a)$, $a \in \mathbb{C}$.

In the last three rings the ideals are precisely the additive subgroups which are clearly principal ideals.

The maximal ideals of $\mathbb{Z}/6$ are thus (2) and (3).

$\mathbb{Z}/17$ is a field, so 0 is the only maximal ideal.

The maximal ideals of $\mathbb{Z}/2008$ are (2) and (251).