Rings and Fields
Test 3.

12 March 2008

1. **10 marks** Find all prime numbers p such that the polynomial $x^3 + 2x + p$ is irreducible in $\mathbb{Q}$.

By Gauss's lemma irreducibility of a polynomial with integer coefficients over $\mathbb{Q}$ is equivalent to its irreducibility over $\mathbb{Z}$. For a monic polynomial of degree 3 this implies the existence of an integer root. This root divides p, so is ± 1 or $\pm p$. Checking these four cases we conclude that the polynomial is irreducible for all primes $p \neq 3$. If $p = 3$ then $x = -1$ is a root.

2. **10 marks** Find all the subfields of $\mathbb{Q}(\sqrt[7]{2})$ (other than this field itself).

The polynomial $x^7 - 2$ is irreducible (e.g. by Eisenstein). Hence it is the minimal polynomial of $\sqrt[7]{2}$, hence $[\mathbb{Q}(\sqrt[7]{2}) : \mathbb{Q}] = 7$. If F is a subfield of $\mathbb{Q}(\sqrt[7]{2})$, then F contains $\mathbb{Q}$, since $\mathbb{Q}$ is contained in every field of characteristic zero (proved in lectures). By a result from lectures, $[F : \mathbb{Q}]$ divides $[\mathbb{Q}(\sqrt[7]{2}) : \mathbb{Q}] = 7$, and so is either 1 or 7. Thus the only subfield of $\mathbb{Q}(\sqrt[7]{2})$ not equal to the field itself is $\mathbb{Q}$.