M2P4 Rings and Fields
Problem Sheet 1.

1. Let R be the ring of 2×2 matrices over $\mathbb{Z}$. Which of the following subsets S are subrings of R? If S is a subring, then determine whether or not S is commutative.

(1) $\left\{ \begin{pmatrix} r & 0 \\ s & t \end{pmatrix} : r \in 2\mathbb{Z},\ s \in 3\mathbb{Z},\ t \in \mathbb{Z} \right\}$;

(2) $\{M \in R : \det M \in 2\mathbb{Z}\}$;

(3) $\left\{ M \in R : \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} M = M \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$;

(4) The set of $M \in R$ such that the column vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of M.

2. Let $m = 5^6$ and $n = 6^5$.

(1) Which elements of $\mathbb{Z}/m$ are zero divisors?

(2) Do the zero divisors of $\mathbb{Z}/m$, together with $\overline{0}$, form a subring of $\mathbb{Z}/m$?

(3) Do the zero divisors of $\mathbb{Z}/n$, together with $\overline{0}$, form a subring of $\mathbb{Z}/n$?

3. (1) Construct an infinite sequence $p_1, p_2, \ldots$ in $\mathbb{Q}[x]$ such that for all i we have $p_i \mid p_{i+1}$ but $p_{i+1} \nmid p_i$.

(2) Construct an infinite sequence of distinct elements $p_1, p_2, \ldots$ in $\mathbb{Q}[x]$ such that for all i we have $p_i \mid p_{i+1}$ and $p_{i+1} \mid p_i$.

(3) Show that there is no infinite sequence $p_1, p_2, \ldots$ in $\mathbb{Q}[x]$ such that for all i we have $p_{i+1} \mid p_i$ but $p_i \nmid p_{i+1}$.

4. Let $R = \mathbb{Z}/15$. Find explicitly all elements a of R such that $aR = R$.

5. Find polynomials $q(x)$ and $r(x)$ such that

$$x^5 - x^2 + x + 2 = (x^2 - x + 1)q(x) + r(x)$$

and $r(x)$ has degree less than 2.

6. Prove that the element $1 + 5i$ of $\mathbb{Z}[i]$ is not irreducible.