# M2P4 Rings and Fields
# Answers Sheet 1.

1. In (1), (3) and (4) we have subrings. (Check that the zero matrix belongs to $S$ and that if $a, b \in S$ then $a + b$, $ab$, $-a \in S$.)

The set $S$ in (2) is not a subring since the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in $S$, but their sum is not.

The ring in (1) is not commutative as the matrices $\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ do not commute.

The ring in (3) is $\left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} : s, t \in \mathbb{Z} \right\}$, and so it is commutative.

The ring in (4) is not commutative, since $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ do not commute.

2. (1) The elements are those $\overline{a}$ such that $5 \mid a$ (and $\overline{a} \neq \overline{0}$).
(2) Yes, since $\overline{0} \in S$ and if $\overline{a}, \overline{b} \in S$ then $\overline{a+b}$, $\overline{ab}$, $\overline{-a} \in S$.
(3) No, since $\overline{3}$ and $\overline{2}$ are zero divisors, but $\overline{1} = \overline{3} - \overline{2}$ is not.

3. (1) Let $p_i = x^i$.
(2) Let $p_i = 2^i$.
(3) If $p_{i+1} \mid p_i$ and $p_i \nmid p_{i+1}$ then $\deg p_{i+1} < \deg p_i$. Hence such a sequence is finite.

4. The elements are $\overline{b}$ where $b = 1, 2, 4, 7, 8, 11, 13$ or $14$. (Take $b$ coprime to 15.)

5. The polynomials are $q(x) = x^3 + x^2 - 2$ and $r(x) = -x + 4$.

6. We have $1 + 5i = (1 + i)(3 + 2i)$ and neither $1 + i$ nor $3 + 2i$ is a unit.