M2P4 Rings and Fields
Problem Sheet 2.

1. Prove that $\mathbb{Q}$ contains infinitely many subrings which are integral domains.

2. Let $f(x) \in \mathbb{C}[x]$ and $\alpha \in \mathbb{C}$. Prove that the remainder, upon dividing $f(x)$ by $x - \alpha$, equals $f(\alpha)$.
(1) For which value of a_0 is $a_0 + 3x - 2x^2 + x^3$ divisible by $x + 1$?
(2) For which values of the natural number n is $x^n - 1$ divisible by $x^2 + 1$?

3. Let $\mathbb{H}$ be the set of 2×2 matrices which is given by

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}.$$

Prove that $\mathbb{H}$ is a ring with a 1, in which each non-zero element has an inverse in $\mathbb{H}$.

Find infinitely many elements r in $\mathbb{H}$ which satisfy $r^2 + 1 = 0$.

Now, we know that there is a theorem which implies that the polynomial $x^2 + 1 \in F[x]$, where F is a field, has at most 2 roots. Does this contradict the result of the last paragraph?

4. Prove that 3 is an irreducible element of $\mathbb{Z}[i]$. Is 13 irreducible?

5. Show that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ if and only if $f(\alpha) = 0$ for some $f(x) \in \mathbb{Z}[x]$.

6. Let $p_0, p_1, p_2, \ldots$ be the prime numbers, in ascending order. Prove that

$$\theta : a_0 + a_1 x + \ldots + a_n x^n \longmapsto p_0^{a_0} p_1^{a_1} \ldots p_n^{a_n}$$

gives a bijection between $\mathbb{Z}[x]$ and the set of positive rational numbers. Deduce that the set of algebraic numbers over $\mathbb{Q}$ is countable. Prove that there exists a complex number which is not algebraic over $\mathbb{Q}$.