M2P4 Rings and Fields
Problem Sheet 3.

1. Suppose that a and b are elements of the commutative ring R. We say that the element d of R is a greatest common divisor of a and b if

(i) $d \mid a$ and $d \mid b$;

(ii) whenever $c \mid a$ and $c \mid b$, we have $c \mid d$.

(1) There are two greatest common divisors of the integers 69290 and 89544. What are they?

(2) Find a greatest common divisor of x^3-2x^2-5x-2 and x^3-x^2-7x-5 in $\mathbb{Q}[x]$.

(3) Find a greatest common divisor of $3+i$ and $4-6i$ in $\mathbb{Z}[i]$.

A greatest common divisor is sometimes called a highest common factor. One way of tackling this problem is to use the Euclidean algorithm, as in M1F.

2. Let $\omega = e^{2\pi i/3}$ and $R = \{m + n\omega : m, n \in \mathbb{Z}\}$.

(1) Prove that R is a subring of $\mathbb{C}$.

(2) Define $\varphi(m+n\omega) = m^2 - mn + n^2$. Prove that $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$ for all $r_1, r_2 \in R$.

(3) Mark the elements of R on the Argand diagram.

(4) Prove that R is a Euclidean domain with norm φ.

3. Let R be as in Question 2. Find a subring of R which is an integral domain but not a Euclidean domain.

4. Prove that every field is a Euclidean domain.

5. Prove that for all n

$$\{a + ib \in \mathbb{Z}[i] : |a + ib| \leq n\}$$

is finite.

Find a Euclidean domain in which there are infinitely many elements of a given norm.