M2P4 Rings and Fields
Answers Sheet 3.

1. (1) The greatest common divisors are 1066 and -1066.

(2) We have

$$\begin{aligned} x^3 - 2x^2 - 5x - 2 &= (x+1)(x^2 - 3x - 2) \\ x^3 - x^2 - 7x - 5 &= (x+1)(x^2 - 2x - 5). \end{aligned}$$

Only units divide both $x^2 - 3x - 2$ and $x^2 - 2x - 5$. Hence $x + 1$ is a g.c.d.

(3) We have

$$\begin{aligned} 3 + i &= (1+i)(2-i) \\ 4 - 6i &= (1+i)(2-3i)(1-i). \end{aligned}$$

All the factors here are irreducible (consider their norms). Hence $1 + i$ is a g.c.d. (and so are $1 - i, -1 + i$ and $-1 - i$).

2. (1) This is easy. (Note that $\omega^2 = -1 - \omega$.)

(2) Either prove this directly, or observe that φ is the square of the usual complex norm.

(3)

(The edges of all the small triangles have length 1.)

(4) Assume that $a \in R$ and $0 \neq b \in R$. Then $\varphi(b) \geq 1$, so $\varphi(a) \leq \varphi(a)\varphi(b) = \varphi(ab)$.

Now, a/b lies in some triangle of edge 1, so there exists $q \in R$ with $|(a/b) - q|^2 < 1$. Let $r = a - bq$. Then $\varphi(r) = |a - qb|^2 = |b|^2 \, |(a/b) - q|^2 < |b|^2 = \varphi(b)$. Thus, $a = qb + r$ with $\varphi(r) < \varphi(b)$.

3. We have that $\mathbb{Z}[\sqrt{-3}]$ is a subring of R (note that $\omega = (-1 + \sqrt{-3})/2$), but $\mathbb{Z}[\sqrt{-3}]$ is not a UFD (since $4 = 2.2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$), so $\mathbb{Z}[\sqrt{-3}]$ is not a Euclidean domain.

4. Define $\varphi(r) = 1$ for all non-zero r in the field.

5. We see that $a^2 + b^2 \leq n^2$ only if $-n \leq a \leq n$ and $-n \leq b \leq n$. Hence the given set is finite.

On the other hand, $\mathbb{Q}$ is a Euclidean domain with $\varphi(r) = 1$ for all $r \in \mathbb{Q}^*$ (see Question 4).