M2P4 Rings and Fields
Problem Sheet 4.

1. Prove that if F_1 and F_2 are subfields of a field K then $F_1 \cap F_2$ is a subfield of K.

2. Let I and J be ideals in a commutative ring R. Define $I + J$ to be the set of $a + b$ where $a \in I$ and $b \in J$. Prove that $I + J$ is an ideal of R.

3. Let a and b be positive integers whose positive greatest common divisor is d. Prove that, for all sufficiently large integers k, we can write kd as $kd = as + bt$ where s and t are non-negative integers.

4. In the game of Sylver Coinage, the two players alternately name natural numbers, but they are not allowed to name any number which is a sum of previously named ones. Thus, after 3 and 5 have been named, neither player may name

$$3,\ 5,\ 6 = 3 + 3,\ 8 = 3 + 5,\ 9 = 3 + 3 + 3,\ 10 = 5 + 5,\ 11 = 3 + 3 + 5,$$

and so on. The first player who names 1 loses. For example, if Player 1 names 2, then Player 2 can name 3, whereupon, Player 1 is forced to name 1, and loses.

(1) Describe a legal game of one million moves.

(2) Prove that the game is finite. That is, one of the players is eventually forced to name 1.

(3) Who wins (assuming best play) if Player 1 names 4 and Player 2 names 5?

(4) Show that Player 2 can win by responding with 6 if Player 1 begins with 4.

(5) Who wins the game, with best play?