M2P4 Rings and Fields
Answers Sheet 4.

1. Check that $0, 1 \in F_1 \cap F_2$ and if $a, b \in F_1 \cap F_2$ then $a+b, ab, -a \in F_1 \cap F_2$; also, if $0 \neq a \in F_1 \cap F_2$ then $a^{-1} \in F_1 \cap F_2$.

2. Let $a_1, a_2 \in I$ and $b_1, b_2 \in J$ and $r \in R$. Then

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) \in I + J$$

$$-(a_1 + b_1) = -a_1 - b_1 \in I + J$$

$$(a_1 + b_1)r = a_1 r + b_1 r \in I + J$$

since $a_1 r \in I$ and $b_1 r \in J$. Hence, in particular, $(a_1 + b_1)(a_2 + b_2) \in I + J$. Therefore, $I + J$ is an ideal of R.

3. There exist integers m, n with $d = am + bn$. One of m and n must be negative and the other positive. Without loss of generality, $m \leq 0$ and $n > 0$.

Let $a = a'd$. Assume that $k > -(a')^2 m$. Then $k = a'q + r$ for some $q \geq -a'm$ and $0 \leq r < a'$. Therefore,

$$kd = aq + rd = a(q + rm) + b(rn).$$

Now, $q + rm \geq q + a'm \geq 0$ (since $m \leq 0$) and $rn \geq 0$. Therefore, kd can be written in the required form if $k > -(a')^2 m$.

4. (1) The game where the Players name, in turn, $n, n-1, n-2, ..., 1$ is legal.

(2) At any time after the first move, let d be the positive g.c.d. of the named numbers. Only finitely many multiples of d are not expressible as sums of numbers already named (compare Question 3). Thus, after a finite number of moves the g.c.d. must be reduced. Eventually, the g.c.d is 1, whereupon there are only finitely many legal moves left.

(3) The only legal moves after $4, 5$ are $1, 2, 3, 6, 7, 11$. Player 1 wins by naming 11.

(4) After the moves $4, 6$, Player 2 repeatedly makes whichever move of $4k + 1$ and $4k + 3$ has not been played by Player 1, until Player 1 names 2 or 3, when the response is 3 or 2 respectively.

(5) Player 1 wins. It has been proved that 5 is a winning first move (as is any prime number, other than 2 or 3).

For more information, see "Winning Ways, Volume 2" by Berlekamp, Conway and Guy.