M2P4 Rings and Fields
Problem Sheet 6.

1. Suppose that F is a finite field with m elements. Prove that $r^m = r$ for all $r \in F$.

2. Construct a field F with 8 elements. Does F contain a subfield with 4 elements? (Hint: consider the multiplicative group of non-zero elements of F.)

3. Prove that $\frac{2}{9}x^5 + \frac{5}{3}x^4 + x^3 + \frac{1}{3}$ is an irreducible polynomial in $\mathbb{Q}[x]$.

4. Factorize the following polynomials into irreducibles.
(1) $x^7 + 11x^3 - 33x + 22 \in \mathbb{Q}[x]$.
(2) $x^4 + 1 \in \mathbb{R}[x]$.
(3) $x^4 + 1 \in \mathbb{Q}[x]$.
(4) $1 + x + x^2 + x^3 + x^4 + x^5 \in \mathbb{Q}[x]$.
(5) $x^3 - 5 \in \mathbb{Z}/11[x]$.
(6) $x^8 - x \in \mathbb{Z}/2[x]$.
(7) $x^2 + \omega x + \omega^2 \in F[x]$, where $F = \{0, 1, \omega, \omega^2\}$ is the field of 4 elements.

5. Let R be a ring in which $r^2 = r$ for all $r \in R$. Prove that R is commutative.

6. Let R be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define addition and multiplication on R in the usual way. That is, for $f, g \in R$ and $x \in \mathbb{R}$ set $(f + g)(x) = f(x) + g(x)$ and $fg(x) = f(x)g(x)$.
(1) Convince yourself that R is a commutative ring.
(2) Let $I = \{f \in R : f(0) = 0\}$. Prove that I is an ideal of R and that I is a maximal ideal.
(3) For $n \in \mathbb{N}$, let $I_n = \{f \in R : f(x) = 0 \text{ for } |x| > n\}$. Prove that I_n is an ideal of R. Show that $I_1 \subset I_2 \subset I_3 \ldots$.

7. Let R be an integral domain. Recall that $R[x]$ is then an integral domain. Describe the elements of the field of fractions of $R[x]$.