M2P4 Rings and Fields
Answers Sheet 6.

1. The non-zero elements of F form a multiplicative group F^* of order $m-1$. Therefore $r^{m-1} = 1$ for all $r \in F^*$, by Lagrange's Theorem. Hence $r^m = r$ for all $r \in F$.

2. $x^3 + x + 1 \in \mathbb{Z}/2[x]$ is irreducible. Hence $\mathbb{Z}/2[x]/(x^3 + x + 1)\mathbb{Z}/2[x]$ is a field of 8 elements.

All the non-zero elements of a field of 4 elements satisfy $r^3 = 1$. But no element r of F, except $r = 1$, satisfies $r^3 = 1$ (since the multiplicative group of F has order 7). Therefore, F does not contain a subfield with 4 elements.

3. The given polynomial is irreducible if and only if $2x^5 + 15x^4 + 9x^3 + 3$ is irreducible, which is the case, by Eisenstein (with $p = 3$).

4. (1) Irreducible, by Eisenstein.
(2) $(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.
(3) Irreducible, by part (2).
(4) $(1 + x)(1 + x + x^2)(1 - x + x^2)$.
(5) $(x - 3)(x^2 + 3x + 9)$.
(6) $x(x + 1)(x^3 + x + 1)(x^3 + x^2 + 1)$.
(7) $(x + 1)(x + \omega^2)$.

5. (1) Easy. Recall that the sum and product of continuous functions are continuous.

(2) It is easy to check that I is an ideal. If $I \subset J$ then take $g \in J$ with $g \notin I$. We have $g(0) = \alpha \neq 0$ and $\alpha - g \in I \subset J$, so $\alpha \in J$. Thus, for all $f \in R$, we have $f = \alpha(\alpha^{-1}f) \in J$ so $J = R$.

(3) Easy.

6. The elements have the form $f(x)/g(x)$ where $f(x), g(x) \in R[x]$ and $g(x) \neq 0$.