M2P4 Rings and Fields
Answers Sheet 7.

1. If $n = ab$ then

$$1 + x + x^2 + ... + x^{n-1} = (1 + x + x^2 + ... + x^{a-1})(1 + x^a + x^{2a} + ... + x^{(b-1)a}).$$

2. $x^3 + nx^2 - 4$ is not irreducible if and only if it has an integer root (by Gauss's Lemma). The root must divide 4, so it is $\pm 1, \pm 2$ or ± 4. Hence the polynomial is irreducible if and only if $n \neq -1, 3, 5$.

3. Put $x = y+1$ to get $y^4 + a'y^3 + b'y^2 + c'y + (2+a+b+c)$, where a', b', c' are even and $2 + a + b + c \equiv 2 \bmod 4$. Hence the polynomial is irreducible, by Eisenstein.

4. Let $\theta = \pi/8$. Since $2^{-\frac{1}{2}} = \sin 2\theta = 2\sin\theta\sqrt{1 - \sin^2\theta}$ we have $8\sin^4\theta - 8\sin^2\theta + 1 = 0$. Put $y = 2x$ in the polynomial $16x^4 - 16x^2 + 2$ to obtain $y^4 - 4y^2 + 2$; hence the polynomial is irreducible, by Eisenstein. It follows that $x^4 - x^2 + \frac{1}{8}$ is the minimal polynomial of $\sin\theta$.

5. $x^3 - 4$. This is irreducible, by Gauss's Lemma, since it has no integer root.

6. $(2 + 5i)(2 - 5i) = 29$. Hence $29 \in I$ and the characteristic is 29.

7. The minimal polynomial of $2^{\frac{1}{5}}$ is $x^5 - 2$, so $|\mathbb{Q}(2^{\frac{1}{5}}) : \mathbb{Q}| = 5$. Now, $|\mathbb{Q}(\alpha) : \mathbb{Q}|$ divides 5, so the minimal polynomial of α has degree 1 or 5.

8. $x^2 - 2$ is irreducible over $\mathbb{Z}_{13}$, so $\mathbb{Z}_{13}[x]/I$ is a field of 169 elements, where $I = (x^2 - 2)\mathbb{Z}_{13}[x]$. (The squares modulo 13 are $1, 3, 4, 9, 10, 12$; hence $x^2 - a$ will work if $a = 2, 5, 6, 7, 8$ or 11.)

9. Construct an equilateral triangle OAB; draw a circle, centre O and radius OA; bisect angle AOB to meet the circle at C; mark off lengths equal to AC around the circle.

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10. Bisect XP at Q; draw a circle, centre Q and radius PQ to intersect the circle C at T. Then PT is the required tangent.